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# Classical Methods for Partial Differential Equations

Exercise sheet 1

**Exercise 1**

Determine the order of the following PDE's. Check if they are linear or non-linear. Find a $F$ as in the lecture with $F\left(x, u_{x_i}, \dots, u_{x_i x_j}, \dots, etc.\right) = 0$ if $u$ solves the PDE.

1. Schrödinger $\mathtt{i}\hbar u_t + \frac{\hbar^2}{2m} \triangle u = V(x,y,z)\, u$, where the constants $\hbar$, $m$ and the function $V \colon \mathbb{R}^3 \to \mathbb{R}$ are given. This equation models the wave function of an electron moving in the potential $V$. $\hbar$ is the normalized Planck's constant and $m$ is the electron's mass.

2. biharmonic $\triangle^2 u = f(x,y)$, where $f \colon \mathbb{R}^2 \to \mathbb{R}$ is given and $\triangle^2 v = \triangle(\triangle v) = \sum_{i,j=1}^{2} \frac{\partial^2}{\partial x_i^2} \frac{\partial^2}{\partial x_j^2} v = \frac{\partial^4 v}{\partial x^4} + 2\frac{\partial^4 v}{\partial x^2 \partial y^2} + \frac{\partial^4 v}{\partial y^4}$ ($v \colon \mathbb{R}^2 \to \mathbb{R}$). This equation models deflection of a thin plate, where $f$ is the load.

3. eikonal $u_x^2 + u_y^2 = n(x,y)^2$, where $n \colon \mathbb{R}^2 \to \mathbb{R}$ is given. This equation models the wave fronts in two dimensional geometric optics. The function $n$ describes the refractive index of the material at the point $(x,y)$.

4. Navier-Stokes $\rho \overrightarrow{u}_t + \rho\left(\overrightarrow{u} \cdot \nabla\right) \overrightarrow{u} = \mu \triangle \overrightarrow{u} - \nabla p$, $\nabla \cdot \overrightarrow{u} = 0$, where $\mu$, $\rho$ are given constants, $\overrightarrow{u} \cdot \nabla = \sum_{i=1}^{3} u_i \frac{\partial}{\partial x_i}$ and $\overrightarrow{u}$, $p$ are the unknowns. This system models the incompressible fluid flow where $\rho$ is the mass density, $\mu$ is the viscosity coefficient, $\overrightarrow{u}$ is the flow velocity and $p$ is the hydrostatic pressure.

5. Korteweg-de Vries $u_t + 6uu_x + u_{xxx} = 0$. This equation models waves on shallow water surfaces.

6. Black-Scholes $u_t + \frac{1}{2}\sigma^2 x^2 u_{xx} + rxu_x - ru = 0$, where $\sigma$, $r$ are given constants. This equations describes the price of an option. It can be derived from Black–Scholes model.

**Exercise 2**

Let $v, w \colon \mathbb{R}^3 \to \mathbb{R}^3$ be continuously differentiable and let $u$ be twice continuously differentiable. Prove the following identities:

1. $\operatorname{div} \operatorname{grad} u = \triangle u$ ($u \colon \mathbb{R}^3 \to \mathbb{R}$),

2. $\operatorname{div}\operatorname{curl} u = 0$ $(u\colon \mathbb{R}^3 \to \mathbb{R}^3)$,

3. $\operatorname{curl}\operatorname{grad} u = 0$ $(u\colon \mathbb{R}^3 \to \mathbb{R})$,

4. $\operatorname{curl}\operatorname{curl} u = \operatorname{grad}\operatorname{div} u - \triangle u$ $(u\colon \mathbb{R}^3 \to \mathbb{R}^3)$, where $\triangle u = \begin{pmatrix} \triangle u_1 \\ \triangle u_2 \\ \triangle u_3 \end{pmatrix}$, for $u\colon \mathbb{R}^3 \to \mathbb{R}^3$,

5. $\operatorname{div}\,(v \times w) = w \cdot \operatorname{curl} v - v \cdot \operatorname{curl} w$.

**Exercise 3**

Assume that the fields $E, B\colon \mathbb{R}^3 \times [0,\infty) \to \mathbb{R}^3$ are twice continuously differentiable and solve the equations

$$\operatorname{curl} E = -\frac{\partial B}{\partial t},\ \operatorname{curl} B = \frac{\partial E}{\partial t}.$$

Show that $\operatorname{div} E$ and $\operatorname{div} B$ do not depend on $t$.

Conclude that in an electrodynamical system, when all four Maxwell's equations are satisfied, vanishing of the current density $j$ implies that the charge density $\varrho$ does not depend on $t$ ("Time dependent charge density implies a nonzero current").

**Exercise 4**

Assume that the fields $E, B\colon \mathbb{R}^3 \times [0,\infty) \to \mathbb{R}^3$ are twice continuously differentiable and solve the Maxwell's equations

$$\operatorname{curl} E = -\frac{\partial B}{\partial t},\ \operatorname{curl} B = j + \frac{\partial E}{\partial t},\ \operatorname{div} B = 0,\ \operatorname{div} E = \varrho,$$

with charge density $\varrho$ and current density $j$. Prove that the conservation of charge holds, i.e. $\frac{\partial \varrho}{\partial t} + \operatorname{div} j = 0$.

Above exercises will be discussed on 28.10.2015.