Karlsruhe Institute of Technology (KIT)
Institute for Analysis
Prof. Dr. Michael Plum
M.Sc. Piotr Idzik

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Classical Methods for Partial Differential Equations

Exercise sheet 11

Exercise 38

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded Lipschitz domain, $Q = \Omega \times (0, T)$ and $u \in \mathcal{C}^2\big(\overline{Q}\big)$ satisfy

$$\frac{\partial u}{\partial t} - \triangle u \leqslant 0 \text{ in } Q.$$

Give an alternative proof of the maximum principle

$$\max_{\overline{Q}} u = \max_{\partial' Q} u,$$

using the *energy method*: Let $M = \max_{\partial' Q} u$, $\psi(y) = \max\{y - M, 0\}^4 \, (y \in \mathbb{R})$. Prove the following statements

1. $\psi(u) = \psi \circ u \in \mathcal{C}^2\big(\overline{Q}\big)$,
2. $\frac{\partial \psi(u)}{\partial t} - \triangle(\psi(u)) \leqslant 0$ in Q,
3. the mapping $t \longmapsto \int_\Omega \psi(u(x,t)) \, \mathrm{d}x$ is monotonically decreasing,
4. $u(x,t) \leqslant M \, ((x,t) \in Q)$.

Can the assumption $u \in \mathcal{C}^2\big(\overline{Q}\big)$ be weakened?

Exercise 39

Consider the inhomogeneous heat equation

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) - \triangle u(x,t) = w(x,t) & (x \in \mathbb{R}^n, t \in (0,T)), \\ u(x,0) = \varphi(x) & (x \in \mathbb{R}^n), \end{cases}$$

where $\varphi \in \mathcal{C}(\mathbb{R}^n)$, $T > 0$ fulfil the conditions from the Theorem 4.1 and $w \in \mathcal{C}(\mathbb{R}^n \times (0,T))$.

1. Apply Duhamel's principle to the inhomogeneous heat equation. Under which assumptions on the function w does this method actually give a solution?

2. Determine a solution of the inhomogeneous heat equation in the case of $n = 1$ for $w(x,t) = x^2 t^2$, $\varphi(x) = x^2$.

Exercise 40

Let $g \in \mathcal{C}^1([a,b])$ be such that $g'(x) < 0\,(x \in [a,b])$. Consider the curve

$$\Gamma = \{(x, g(x)) : x \in (a,b)\},$$

and the initial value problem

$$\begin{cases} \frac{\partial^2 u}{\partial x \partial y}(x,y) &= r(x,y) \quad ((x,y) \in (a,b) \times (g(b)\,, g(a)))\,, \\ u(x, g(x)) &= U(x) \quad (x \in (a,b))\,, \\ \frac{\partial u}{\partial x}(x, g(x)) &= U_1(x) \quad (x \in (a,b))\,, \\ \frac{\partial u}{\partial y}(x, g(x)) &= U_2(x) \quad (x \in (a,b))\,, \end{cases}$$

where $r \in \mathcal{C}(\mathbb{R}^2)$, $U \in \mathcal{C}^1([a,b])$, $U_1, U_2 \in \mathcal{C}([a,b])$ and $U'(x) = U_1(x) + g'(x)\,U_2(x)\,(x \in [a,b])$. Show that the above initial value problem has an unique solution and find its integral representation.

Exercise 41

Let $\Omega \subseteq \mathbb{R}^2$ be a domain. Consider the following differential equation

$$a\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) \frac{\partial^2 u}{\partial x^2} + 2b\left(x, y, u, \frac{\partial u}{\partial x} \frac{\partial u}{\partial y}\right) \frac{\partial^2 u}{\partial x \partial y} + c\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) \frac{\partial^2 u}{\partial y^2} = d\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) \quad ((x,y) \in \Omega)$$

with the following initial values, for all $t \in (t_0, t_1)$

$$u(x(t)\,, y(t)) = U(t)\,, \; \frac{\partial u}{\partial x}(x(t)\,, y(t)) = U_1(t)\,, \; \frac{\partial u}{\partial y}(x(t)\,, y(t)) = U_2(t)\,,$$

where $a, b, c, d \in \mathcal{C}^k(\Omega \times \mathbb{R}^3)$ and the curve $\Gamma = \{(x(t)\,, y(t)) : t \in (t_0, t_1)\} \subseteq \Omega$ is non-characteristic, with x, y in $\mathcal{C}^1((t_0, t_1))$ Let $u \in \mathcal{C}^k(\Omega)$ be a solution of the above initial value problem. Show that the derivatives of u up to order k on Γ can be determined by the data U, U_1, U_2, x, y and a, b, c, d, and their derivatives (without knowing the solution u explicitly).

Above exercises will be discussed on 20.01.2016.