Exercise 42

Determine the type and find all characteristics of the following differential equations

1. \( y \frac{\partial^2 u}{\partial x^2}(x, y) + \frac{\partial^2 u}{\partial y^2}(x, y) = 0 \) (Tricomi equation),

2. \( \frac{\partial^2 u}{\partial x^2}(x, y) - (1 + y^2) \frac{\partial^2 u}{\partial y^2}(x, y) = 0 \), Hint: \( \int \frac{1}{\sqrt{1+x^2}} \, dx = \text{arsinh} x \).

Exercise 43

Consider the minimal surface equation on the annulus
\[
\Omega = \left\{ (x, y) \in \mathbb{R}^2 : r_1 < \sqrt{x^2 + y^2} < r_2 \right\}, \quad 0 < r_1 < r_2.
\]

Note that the set \( \Omega \) is not simply connected. Determine all radially symmetric solutions (i.e. depending only on \( r = |x| \)) of this equation. For which \( \varphi_1, \varphi_2 \in \mathbb{R} \) is the corresponding boundary value problem
\[
\begin{align*}
u &= \varphi_1 \text{ on } \Gamma_1 = \left\{(x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} = r_1 \right\}, \\
u &= \varphi_1 \text{ on } \Gamma_2 = \left\{(x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} = r_2 \right\},
\end{align*}
\]
solvable?

Exercise 44

Let \( \Omega \subseteq \mathbb{R}^n \) be a bounded and simply connected Lipschitz domain and let \( r \in C(\overline{\Omega}, \mathbb{R}) \). Analogously as in the lecture, derive the Euler–Lagrange equations for the following variational problems:

1. minimize \( J[u] = \int_{\Omega} \left( \frac{1}{2} |\nabla u|^2 - ru \right) \, dx \) on the set \( \{ u \in C^2(\overline{\Omega}) : u = 0 \text{ on } \partial \Omega \} \),

2. minimize \( J[u] = \int_{\Omega} \left( \frac{1}{2} |\nabla u|^2 - ru \right) \, dx \) on the set \( C^2(\overline{\Omega}) \).

Above exercises will be discussed on 27.01.2016.