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Classical Methods for Partial Differential Equations

Exercise sheet 12

Exercise 42

Determine the type and find all characteristics of the following differential equations

1. $y\frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) = 0$ (Tricomi equation),
2. $\frac{\partial^2 u}{\partial x^2}(x,y) - (1+y^2)\,\frac{\partial^2 u}{\partial y^2}(x,y) = 0$, *Hint:* $\int \frac{1}{\sqrt{1+x^2}}\,\mathrm{d}x = \operatorname{arsinh} x$.

Exercise 43

Consider the minimal surface equation on the annulus

$$\Omega = \left\{(x,y) \in \mathbb{R}^2 \colon r_1 < \sqrt{x^2+y^2} < r_2\right\},\ 0 < r_1 < r_2.$$

Note that the set Ω is not simply connected. Determine all radially symmetric solutions (i.e. depending only on $r = |x|$) of this equation. For which $\varphi_1, \varphi_2 \in \mathbb{R}$ is the corresponding boundary value problem

$$\begin{aligned} u = \varphi_1 \text{ on } \Gamma_1 &= \left\{(x,y) \in \mathbb{R}^2 \colon \sqrt{x^2+y^2} = r_1\right\}, \\ u = \varphi_1 \text{ on } \Gamma_2 &= \left\{(x,y) \in \mathbb{R}^2 \colon \sqrt{x^2+y^2} = r_2\right\}, \end{aligned}$$

solvable?

Exercise 44

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded and simply connected Lipschitz domain and let $r \in \mathcal{C}\left(\overline{\Omega}, \mathbb{R}\right)$. Analogously as in the lecture, derive the Euler–Lagrange equations for the following variational problems:

1. minimize $J[u] = \int_\Omega \left(\frac{1}{2}\left|\nabla u\right|^2 - ru\right)\,\mathrm{d}x$ on the set $\left\{u \in \mathcal{C}^2\left(\overline{\Omega}\right) : u = 0 \text{ on } \partial\Omega\right\}$,
2. minimize $J[u] = \int_\Omega \left(\frac{1}{2}\left|\nabla u\right|^2 - ru\right)\,\mathrm{d}x$ on the set $\mathcal{C}^2\left(\overline{\Omega}\right)$.

Above exercises will be discussed on 27.01.2016.