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Classical Methods for Partial Differential Equations

Exercise sheet 14

Exercise 48

Let $\Omega = (0,\infty)^2$.

1. Determine the type of the following partial differential equation, reduce it to its normal form and obtain its general solution
$$x^2\frac{\partial^2 u}{\partial x^2}(x,y) + 2xy\frac{\partial^2 u}{\partial x \partial y}(x,y) + y^2\frac{\partial u^2}{\partial y^2}(x,y) = 4x^2 \quad ((x,y)\in\Omega)\,.$$
2. Write the equation
$$x^2\frac{\partial^2 u}{\partial x^2}(x,y) + 2xy\frac{\partial^2 u}{\partial x \partial y}(x,y) + y^2\frac{\partial u^2}{\partial y^2}(x,y) = 4x^2 + x^4\frac{\partial u}{\partial x} \quad ((x,y)\in\Omega)\,.$$
in the form
$$\frac{\partial^2 u}{\partial \xi^2} = \frac{\partial u}{\partial \eta} + \gamma(\xi,\eta,u)\,.$$

Exercise 49

Prove the theorem 7.1 in the stated generality. *Hint:* Use a weight function:
$$e^{K(|x-h(y)|+|y-g(x)|)}.$$

Exercise 50

Let $g \in \mathcal{C}^1([a,b])$ be such that $g(a)=0$ and $g'(x)>0\,(x\in[a,b])$. Moreover let $\Omega = \{(x,y)\in\mathbb{R}^2 \colon a<x<b, 0<y<g(x)\}$. Consider the initial value problem
$$\begin{cases} \frac{\partial^2 u}{\partial x\partial y}(x,y) &= d\Big(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\Big) & ((x,y)\in\Omega)\,,\\ u(x,0) &= \varphi(x) & (x\in[a,b])\,,\\ u(x,g(x)) &= \psi(x) & (x\in[a,b])\,, \end{cases}$$
where $d\colon \overline{\Omega}\times\overline{\Omega}\times\mathbb{R}^3 \to \mathbb{R}$ is continuous and satisfies the Lipschitz condition of the form
$$|d(x,y,u,p,q) - d(x,y,\widetilde{u},\widetilde{p},\widetilde{q})| \leqslant L(|u-\widetilde{u}| + |p-\widetilde{p}| + |q-\widetilde{q}|)\,,$$

for some constant $L > 0$. Assume that $\varphi, \psi \in \mathcal{C}^1([a,b])$ and $\varphi(a) = \psi(a)$.

Are the curves

$$\Gamma_1 = \left\{(x,0) \in \mathbb{R}^2 \colon x \in [a,b]\right\}, \Gamma_2 = \left\{(x,g(x)) \in \mathbb{R}^2 \colon x \in [a,b]\right\},$$

non-characteristic?

Show that under these assumptions the above problem has an unique solution u satisfying $u \in \mathcal{C}\left(\overline{\Omega}\right) \cap \mathcal{C}^1\left(\overline{\Omega}\right)$ and $\frac{\partial^2 u}{\partial x \partial y} \in \mathcal{C}(\Omega)$.

Exercise 51

Consider the differential equation

$$a\frac{\partial^2 u}{\partial x^2} + 2b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2}{\partial y^2} - \frac{\partial^2 u}{\partial z^2} = 0,$$

with $a, b, c \in \mathbb{R}$, $a > 0$, $c > 0$, $ac > b^2$. Let

$$M = \left\{(x,y,z) \in \mathbb{R}^3 \colon \varphi(x,y,z) = (z-1)^2 - x^2 - y^2 = 0\right\} \setminus (0,0,1).$$

1. Determine the characteristic directions.
2. Let $b = 0$. Which points of M are characteristic?
3. For $a = 2$, $b = 0$, $c = 1$ transform the differential equation according to the change of coordinates $\xi = x$, $\eta = y$, $t = \varphi(x,y,z)$.

Above exercises will be discussed on 10.02.2016.