Classical Methods for Partial Differential Equations
Exercise sheet 2

Exercise 5

Let $\Omega \subseteq \mathbb{R}^3$ be star shaped, i.e. there exists a point $x_0 \in \Omega$ such that for every point $x \in \Omega$ and every $t \in [0,1]$, the point $x_0 + t(x - x_0) \in \Omega$. The point $x_0$ is then called a centre of the set $\Omega$. Moreover assume that the set $\Omega$ is open.

1. (existence of a scalar potential) Let $v \in C^1(\Omega, \mathbb{R}^3)$ be such that $\text{curl} \, v = 0$. Show that the function $\Phi: \Omega \to \mathbb{R}$ defined as

$$\Phi(x) = \int_0^1 (x - x_0) \cdot v(x_0 + t(x - x_0)) \, dt \quad (x \in \Omega),$$

is continuously differentiable, and $\text{grad} \, \Phi = v$ in $\Omega$.

2. (existence of a vector potential) Let $w \in C^1(\Omega, \mathbb{R}^3)$ be such that $\text{div} \, w = 0$. Show that the function $A: \Omega \to \mathbb{R}^3$ defined as

$$A(x) = \left( \int_0^1 tw(x_0 + t(x - x_0)) \, dt \right) \times (x - x_0) \quad (x \in \Omega),$$

is continuously differentiable, and $\text{curl} \, A = w$ in $\Omega$.

Exercise 6

Let $u \in C^1(\mathbb{R}^3, \mathbb{R}^3)$. Assume that there exists a constant $c > 0$ such that

$$|u(x)| \leq \frac{c}{|x|^3 + 1} \quad (x \in \mathbb{R}^3),$$

and that $\int_{\mathbb{R}^3} |\text{div} \, u(x)| \, dx < \infty$. Show that

$$\int_{\mathbb{R}^3} (\text{div} \, u)(x) \, dx = 0.$$

Hint: Integrate over the ball $B_R(0)$ and take $R \to \infty$.

Exercise 7

Let $T \in \mathbb{R}^{n \times n}$ be an orthogonal matrix and let $u \in C^2(\mathbb{R}^n, \mathbb{R})$. Prove that

$$\left( \Delta (u \circ T) \right)(x) = (\Delta u)(Tx) \quad (x \in \mathbb{R}^n).$$
Exercise 8

Consider the incompressible, stationary Euler equation

\[ \rho (\vec{v} \cdot \nabla) \vec{v} = -\nabla p, \quad \nabla \cdot \vec{v} = 0, \]

where the mass density \( \rho > 0 \) of the fluid is assumed to be constant, \( \vec{v} : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) is the flow velocity vector field, and \( p : \mathbb{R}^3 \rightarrow \mathbb{R} \) is the hydrostatic pressure. Moreover, let the flow be irrotational, i.e. \( \text{curl} \; \vec{v} = 0 \). Show that

\[ \frac{1}{2} \rho |\vec{v}|^2 + p, \]

is constant (Bernoulli’s theorem: ”the sum of dynamic pressure and hydrostatic pressure is constant”). \textit{Hint:} Show first that \( \frac{1}{2} \nabla (|\vec{v}|^2) = (\vec{v} \cdot \nabla) \vec{v} + \vec{v} \times \text{curl} \; \vec{v} \).

Above exercises will be discussed on 04.11.2015.

http://www.math.kit.edu/iana2/edu/classicmethopde2015w/