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Classical Methods for Partial Differential Equations

Exercise sheet 3

Exercise 9

Let u be a solution of the n-dimensional wave equation $\frac{\partial^2 u}{\partial t^2} - \triangle u = 0$ of the form $u(x,t) = v(|x|, t)$, where $v \in \mathcal{C}^2([0,\infty) \times [0,\infty), \mathbb{R})$. Such u is called then *a spherical wave*.

1. Show that the function $v = v(r,t)$ satisfies the following equation
$$\frac{\partial^2 v}{\partial t^2} = \frac{\partial^2 v}{\partial r^2} + \frac{n-1}{r}\frac{\partial v}{\partial r}.$$
2. Derive a differential equation for the function $U(r,t) = v(r,t)\, r^{\frac{n-1}{2}}$.
3. In case $n = 3$ derive a general formula for spherical waves.
4. In the case $n = 3$ find, for $t < r$, a formula for the spherical wave satisfying the following initial conditions
$$u(r,0) = u_0(r)\,,\ \frac{\partial u}{\partial t}(r,0) = u_1(r) \quad (r \in (0,\infty))\,,$$
where $u_0, u_1 \in \mathcal{C}^2((0,\infty))$ are given.
5. Assume that $u_i \in \mathcal{C}^2([0,\infty))$ and $u_i'(0) = 0\,(i = 0,1)$. Find a solution of the above problem for all $r > 0$ and $t \geqslant 0$. Show that u remains bounded for $r \to 0$.

Exercise 10

Standing waves are the solutions of the wave equation having the form $u(x,t) = v(x)\sin(\omega t + \varphi)$ where $\omega > 0$, $\varphi \in [0, 2\pi)$ and $v \in \mathcal{C}^2(\mathbb{R}^n)$.

1. Derive a differential equation for the function v.
2. Let $n = 1$. Determine all standing waves. Represent each standing wave as a superposition of two waves, one travelling to the left and one to the right.
3. In case of $n = 3$ find all standing spherical waves, that is all solutions of the wave equation having the form $u(x,t) = v(|x|)\sin(\omega t + \varphi)$, where $v \in \mathcal{C}^2((0,\infty))$.

Exercise 11

Let u be a solution of a one dimensional wave equation with initial data $u(x,0) = u_0(x)$ and $\frac{\partial u}{\partial t}(x,0) = u_1(x)$ $(x \in \mathbb{R})$. Assume that $u_0 \in \mathcal{C}^2(\mathbb{R})$, $u_1 \in \mathcal{C}^1(\mathbb{R})$ and that there exists an $R > 0$ such that $u_0(0) = u_1(0) = 0$ $(|x| > R)$. Show that there exists a $t_0 > 0$ such that

$$\frac{1}{2}\int_{\mathbb{R}} \left(\frac{\partial u}{\partial t}(x,t)\right)^2 \,\mathrm{d}x = \frac{1}{2}\int_{\mathbb{R}} \left(\frac{\partial u}{\partial x}\right)^2 \,\mathrm{d}x \quad (t > t_0)\,,$$

i.e. for all times $t > t_0$ kinetic and potential energies are equal.

Above exercises will be discussed on 11.11.2015.