

Classical Methods for Partial Differential Equations

Exercise sheet 4

Exercise 12

Solve the following initial value problem for the three dimensional homogeneous wave equation

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \Delta u = 0 & \text{in } \mathbb{R}^3 \times (0, \infty), \\ u(x, 0) = 0 & (x \in \mathbb{R}^3), \\ \frac{\partial u}{\partial t}(x, 0) = x_1^2 + x_1 x_2 + x_3^3 & (x = (x_1, x_2, x_3) \in \mathbb{R}^3). \end{cases}$$

In the following exercise we will use the notation

$$D^\alpha u = \frac{\partial^{|\alpha|}}{\partial \hat{x}_1^{\alpha_1} \dots \partial \hat{x}_n^{\alpha_n}},$$

where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$, $|\alpha| = \alpha_1 + \dots + \alpha_n$ and $\hat{x} = (\hat{x}_1, \dots, \hat{x}_n) = (x_1, \dots, x_{n-1}, t)$. Moreover we use the convention that $\mathbb{R}^3 \times [0, \infty) \subseteq \mathbb{R}^4$.

Exercise 13

Let $u \in \mathcal{C}^2(\mathbb{R}^3 \times [0, \infty))$ be a solution of the three dimensional homogeneous wave equation and suppose that $U(0) < \infty$, where

$$U(t) = \sum_{|\alpha| \leq 2} \int_{\mathbb{R}^3} |D^\alpha u(x, t)| \, dx.$$

1. Show that there exists a constant $K > 0$ such that

$$|u(x, t)| \leq \frac{K}{t} U(0) \quad (x \in \mathbb{R}^3, t \geq 1).$$

Hint: Let $S = \{y \in \mathbb{R}^3 : |y| = 1\}$. Write the solution as a integral of the form $\int_S f \cdot \nu \, d\sigma$ with a suitable function f and apply the divergence theorem.

2. Show that if $\lim_{t \rightarrow \infty} \frac{U(t)}{t} = 0$, then $u(x, t) = 0$ ($(x, t) \in \mathbb{R}^3 \times [0, \infty)$). *Hint:* Apply the above result for the function $v(x, t) = u(x, T - t)$ ($T > 0$ sufficiently large).

Exercise 14

Let $u \in \mathcal{C}^2(\mathbb{R}^n \times [0, \infty))$, $n \leq 3$ be a solution of the homogeneous wave equation. Suppose that $u(x, 0) = 0$ and $\frac{\partial u}{\partial t}(x, 0) = 0$ for all $x \in \mathbb{R}^n$ such that $|x| > R$, for some constant $R > 0$. Show that the energy integral

$$E(t) = \frac{1}{2} \int_{\mathbb{R}^n} \left(\frac{\partial u}{\partial t}(x, t) \right)^2 + |\nabla u(x, t)|^2 dx,$$

does not depend on time t .

Above exercises will be discussed on 18.11.2015.