Karlsruhe Institute of Technology (KIT)
Institute for Analysis
Prof. Dr. Michael Plum
M.Sc. Piotr Idzik

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Classical Methods for Partial Differential Equations

Exercise sheet 5

Exercise 15

1. Applying Duhamel's principle solve the following initial value problem for the one dimensional wave equation:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} &= w(x,t) \quad \text{in } \mathbb{R} \times [0,\infty)\,, \\ u(x,0) &= u_0(x) \quad (x \in \mathbb{R})\,, \\ \frac{\partial u}{\partial t}(x,0) &= u_1(x) \quad (x \in \mathbb{R})\,. \end{cases}$$

2. Let $w(x,t) = x^2$, $u_0(x) = x$, $u_1(x) = 0$. Find the explicit solution of the above problem.

Exercise 16

Let $\Omega = (0,a) \times (0,b)$ (for some $a, b > 0$), $u_0 \in \mathcal{C}^2\big(\overline{\Omega}\big)$ and $\frac{\partial u_0}{\partial \nu}(0) = 0$ on $\partial\Omega$. Using separation of variables find the solution of the following boundary value problem for the heat equation:

$$\begin{cases} \frac{\partial u}{\partial t} &= \triangle u \quad \text{in } \Omega \times (0,\infty) \\ u(x,0) &= u_0(x) \quad (x \in \Omega) \\ \frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial\Omega \times (0,\infty) \end{cases}$$

What happens with the solution when $t \to \infty$? Give a physical interpretation of this result. *Hint:* Use without proof that the Fourier series $u_0(x_1,x_2) = \sum_{n,m=0}^{\infty} a_{n,m} \cos\big(\frac{n\pi}{a}x_1\big)\cos\big(\frac{m\pi}{b}x_2\big)$ converges uniformly, where

$$a_{n,m} = \frac{c_{n,m}}{ab}\int_{\Omega} u_0(x_1,x_2)\cos\Big(\frac{n\pi}{a}x_1\Big)\cos\Big(\frac{m\pi}{b}x_2\Big)\,\mathrm{d}x_1\,\mathrm{d}x_2 \quad (n,m \in \mathbb{N}_0)\,,$$

and

$$c_{n,m} = \begin{cases} 1 & (n = m = 0)\,, \\ 2 & (n = 0, m \in \mathbb{N})\,, \\ 4 & (n,m \in \mathbb{N})\,. \end{cases}$$

Exercise 17

Let $u \in \mathcal{C}^4(\mathbb{R} \times [0,\infty))$, $c > 0$, $c \neq 1$ and

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)\left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\right)u(x,t) = 0.$$

Show that there exist functions $g_i \in \mathcal{C}^4(\mathbb{R})\,(i \in \{1, \dots, 4\})$ such that

$$u(x,t) = g_1(x+ct) + g_2(x-ct) + g_3(x+t) + g_4(x-t)\,.$$

Above exercises will be discussed on 25.11.2015.