Karlsruhe Institute of Technology (KIT) WS 2015/2016
Institute for Analysis 25.11.2015
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Classical Methods for Partial Differential Equations

Exercise sheet 6

Exercise 18

Let $\Omega = (0,a)\times(0,b)$ for some $a,b>0$. Using separation of variables find the eigenvalues and the eigenfunctions of the following problem (with periodic boundary conditions):

$$\begin{cases} -\triangle u + \frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = \lambda u & \text{in } \Omega, \\ u(0,y) = u(a,y)\,, \quad \frac{\partial u}{\partial x}(0,y) = \frac{\partial u}{\partial x}(a,y) & (y\in(0,b))\,, \\ u(x,0) = u(x,b)\,, \quad \frac{\partial u}{\partial y}(x,0) = \frac{\partial u}{\partial y}(x,b) & (x\in(0,a))\,. \end{cases}$$

Exercise 19

1. In $\mathbb{R}^2$ the polar coordinates are introduced as follows: $x = r\cos\varphi$, $y = r\sin\varphi$. Show that
$$\triangle = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}.$$

2. In $\mathbb{R}^3$ the spherical coordinates can be introduced as follows:
$$x = r\sin\vartheta\cos\varphi,\ y = r\sin\vartheta\sin\varphi,\ z = r\cos\vartheta.$$
Show that
$$\triangle = \frac{1}{r^2}\left(\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\sin\vartheta\frac{\partial}{\partial\vartheta} + \frac{1}{\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2}\right).$$

3. In $\mathbb{R}^3$ the cylindrical coordinates are introduced as follows: $x = \rho\cos\varphi$, $y = \rho\sin\varphi$, $z = z$. Show that
$$\triangle = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial z^2}.$$

Exercise 20

Let $\Omega \subseteq \mathbb{R}^n$ be an open set with Lipschitz boundary, $c \in \mathcal{C}(\overline{\Omega})$ be such that $c(x) \geqslant 0\,(x\in\Omega)$.

1. Show that the Dirichlet boundary value problem
$$\begin{cases} -\triangle u + cu &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases}$$
has only the trivial solution.
2. Show that the Neumann boundary value problem
$$\begin{cases} -\triangle u + cu &= 0 \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial\Omega, \end{cases}$$
has only the trivial solution, if $c(\xi) > 0$ for some $\xi \in \Omega$. What non-trivial solution occurs in the case $c = 0$.
3. Let $\gamma\colon \partial\Omega \to [0, \infty)$ be a continuous function. What can be said about the solution of the Robin boundary value problem
$$\begin{cases} -\triangle u + cu &= 0 \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} + \gamma u &= 0 \quad \text{on } \partial\Omega. \end{cases}$$

Exercise 21

Let $\Omega \subseteq \mathbb{R}^n$ be a Lipschitz domain, $r \in \mathcal{C}\big(\overline{\Omega}\big)$ and $\varphi \in \mathcal{C}(\partial\Omega)$. Show that the equality
$$\int_\Omega r \,\mathrm{d}x = \int_{\partial\Omega} \varphi \,\mathrm{d}\sigma,$$
is a necessary condition for the existence of a solution of the boundary value problem
$$\begin{cases} \triangle u &= r \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} &= \varphi \quad \text{on } \partial\Omega. \end{cases}$$

Above exercises will be discussed on 02.12.2015.

The classes from 23rd of December 2015 (Wednesday, 14:00-15:30) are moved to 21st of December 2015 (Monday, 9:45 - 11:15). They will take place in the room 2.066 Kollegiengebäude Mathematik (20.30).

The exam will take place on 29th of February 2016 (Monday). The room and the exact time will be announced later.