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Classical Methods for Partial Differential Equations

Exercise sheet 7

Exercise 22

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded Lipschitz domain. Show that the Green's function for the Dirichlet boundary value problem for the Poisson equation is unique.

Exercise 23

Let $\Omega \subseteq \mathbb{R}^n$ be a Lipschitz domain and let G be the Green's function for the Dirichlet boundary value problem for the Poisson's equation. Show that

$$G(\xi, x) = G(x, \xi) \quad (x, \xi \in \Omega) .$$

Hint: For $x_1, x_2 \in \Omega$, $x_1 \neq x_2$ consider

$$\int_{\Omega_0} (g_1 \triangle g_2 - g_2 \triangle g_1) \, \mathrm{d}\xi,$$

where $g_i(\xi) = G(x_i, \xi)$ $(i = 1, 2)$ and $\Omega_0 = \Omega \setminus \left(\overline{\mathrm{B}_\delta(x_1)} \cup \overline{\mathrm{B}_\delta(x_2)}\right)$, with $\delta > 0$ such that $\overline{\mathrm{B}_\delta(x_1)} \cup \overline{\mathrm{B}_\delta(x_1)} \subseteq \Omega$ and $\overline{\mathrm{B}_\delta(x_1)} \cap \overline{\mathrm{B}_\delta(x_2)} = \emptyset$.

Exercise 24

Determine the Green's function for the Dirichlet boundary value problem for the Poisson equation on a two dimensional disc of the form $\Omega = \{x \in \mathbb{R}^2 \colon\ |x| < R\}$.

Exercise 25

1. Let $\Omega \subseteq \mathbb{R}^n$ and let $u \in \mathcal{C}(\Omega)$ be given. Moreover assume that $f \colon \mathbb{R} \to \mathbb{R}$ is convex. Show that
   (a) if the function u is harmonic, then $f \circ u$ is subharmonic,
   (b) if, in addition, the function f is monotonically increasing and if u is subharmonic, then $f \circ u$ is subharmonic.
2. For each $\alpha \in \mathbb{R}$ consider a function $x \longmapsto |x|^\alpha$ on $\mathrm{B}_1(0)$, or on $\mathrm{B}_1(0) \setminus \{0\}$ if $\alpha < 0$. Determine for which α the above function is subharmonic, and for which is superharmonic.

Above exercises will be discussed on 09.12.2015.