Karlsruhe Institute of Technology (KIT)
Institute for Analysis
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WS 2015/2016
29.02.2016

Classical Methods for Partial Differential Equations

Exam sheet WS 2015/2016

First name:	
Family name:	
Matriculation number:	

- The exam sheet consists of eight problems. A maximum of 28 points can be achieved. 9 points are required to pass the exam.
- The exam will take 150 minutes.
- Write your name and matriculation number on each of the sheets.
- Do not use pencils and red ink.
- You can write your solutions in English or German.

To be completed by the corrector:

p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	$\sum_{i=1}^{8} p_i$	grade:

Problem 1 (3 points)

Using Kirchoff's formula, solve the following initial value problem for the three dimensional homogeneous wave equation

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \triangle u &= 0 & \text{in } \mathbb{R}^3 \times [0, \infty)\,, \\ u(x,0) &= 0 & (x \in \mathbb{R}^3)\,, \\ \frac{\partial u}{\partial t}(x,0) &= x_1 x_2 + 3x_3 & (x = (x_1, x_2, x_3) \in \mathbb{R}^3)\,. \end{cases}$$

Problem 2 (3 points)

1. Using Duhamel's principle, derive, without using the corresponding result from lecture or exercises, the integral representation of the solution of the following initial value problem for the one dimensional wave equation:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} & = & w(x,t) \quad \text{in } \mathbb{R} \times [0,\infty)\,, \\ u(x,0) & = & u_0(x) \quad (x \in \mathbb{R})\,, \\ \frac{\partial u}{\partial t}(x,0) & = & u_1(x) \quad (x \in \mathbb{R})\,. \end{cases}$$

2. Let $w(x,t) = x$, $u_0(x) = x^2$, $u_1(x) = \sin x$. Find the explicit solution of the above problem.

Problem 3 (3 points)

Let $\Omega = (0, a)$ (for some $a > 0$), $u_0 \in \mathcal{C}^2\left(\overline{\Omega}\right)$ and $u_0'(0) = u_0'(a) = 0$. Using separation of variables find the solution of the following boundary value problem for the heat equation:

$$\begin{cases} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} & \text{in } \Omega \times (0, \infty)\,, \\ \frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(a, t) &= 0 & (t \in (0, \infty))\,, \\ u(x, 0) &= u_0(x) & (x \in \Omega)\,. \end{cases}$$

What happens to the solution when $t \to \infty$? Give a physical interpretation of this result. *Hint:* Use without proof that the Fourier series $\sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi}{a} x\right)$ converges uniformly to u_0, where $a_n = \frac{2}{a} \int_0^a u_0(x) \cos\left(\frac{n\pi}{a} x\right)\, \mathrm{d}x$ ($n \in \mathbb{N}$), and $a_0 = \frac{1}{a} \int_0^a u_0(x)\, \mathrm{d}x$.

Problem 4 (4 points)

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded Lipschitz domain. Without using the corresponding result from lecture or exercises, show that the Green's function for the Dirichlet boundary value problem for the Poisson equation is unique (if it exists).

Problem 5 (4 points)

Let $X \subseteq \mathbb{R}^n$. A point $x \in X$ is called an *isolated point of* X if there exists a ball $\mathrm{B}_r(x)$ such that $(\mathrm{B}_r(x) \setminus \{x\}) \cap X = \emptyset$.

Let $n \geqslant 2$. Show that the set of zeroes of any (real valued) harmonic function does not contain any isolated points. *Hint:* midpoint formula.

Problem 6 (4 points)

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain.

1. Without using the corresponding result from lecture or exercises, show that if Ω is strictly convex, then for each $\xi \in \partial\Omega$ there exists a barrier function.

2. Consider the Dirichlet problem for the Laplace equation

$$\begin{cases} \triangle u &= 0 \quad \text{in } \Omega, \\ u &= f \quad \text{on } \partial\Omega. \end{cases}$$

   Show that if this problem has a solution $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}\left(\overline{\Omega}\right)$ for each continuous function $f \colon \partial\Omega \to \mathbb{R}$, then for each $\xi \in \partial\Omega$ there exists a barrier function.

Problem 7 (2 points)

Let $u_1, \ldots, u_n$ be solutions of the one dimensional heat equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ $(x \in \mathbb{R}, t > 0)$. Without using the corresponding result from lecture or exercises, show that the function

$$u(x,t) = u(x_1, \ldots, x_n, t) = \prod_{i=1}^{n} u_i(x_i, t) \quad (x \in \mathbb{R}^n, t > 0)\,,$$

is a solution of the $n-$dimensional heat equation $\frac{\partial u}{\partial t} = \triangle u$.

Problem 8 (5 points)

Determine the type of the following second order partial differential equation, reduce it to its normal form and obtain its general solution

$$xy\frac{\partial^2 u}{\partial x^2}(x,y) + x^2\frac{\partial^2 u}{\partial x \partial y} - y\frac{\partial u}{\partial x} = 0 \quad (x, y \in (0, \infty)) .$$