Proof of Exercise 30: Let $w : \overline{\Omega} \cap N \to \mathbb{R}$ be the local barrier function. Let $\rho > 0$ s.t. $B_\rho(\xi) \subset N$. We dnote

$$m := \sup_{x \in \left(B_\rho(\xi) \setminus B_{\frac{\rho}{4}}(\xi)\right) \cap \overline{\Omega}} w(x).$$

The definition of w implies that $m < 0$. Let

$$\overline{w}(x) = \begin{cases} \max\{w(x),\, m\},\, x \in \overline{\Omega} \cap B_{\frac{\rho}{4}}(\xi), \\ m,\, x \in \overline{\Omega} \setminus B_{\frac{\rho}{4}}(\xi). \end{cases}$$

For $\forall x \in \left(B_\rho(\xi) \setminus B_{\frac{\rho}{4}}(\xi)\right) \cap \overline{\Omega}$, $w(x) \le m$ implies $\overline{w}(x) = m = \max\{m,\, w(x)\}$. We are going to show that $\overline{w}(x)$ is a barrier function at $\xi \in \partial\Omega$:

(a) The maps $x \mapsto m$ and $x \mapsto w(x)$ are continuous and subharmonic on $\Omega \cap B_\rho(\xi)$, and hence, $\overline{w}$(x) is continuous and subharmonic on $\Omega \cap B_\rho(\xi)$. It's easy to see that $\overline{w}(x) = m$, $\forall x \in \Omega \setminus B_{\frac{\rho}{4}}(\xi)$ is continuous and subharmonic. Hence, $\overline{w}$(x) is continuous and subharmonic on Ω.

(b) $w(\xi) = \{w(\xi),\, m\} = 0$.

(c) Let $x \in \partial\Omega \cap B_\rho(\xi)$, then $w(x) = \max\{w(x),\, m\} < 0$. Let $x \in \partial\Omega \setminus B_\rho(\xi)$, then $w(x) = m < 0$. Hence, $\overline{w}(x) < 0$, $\forall x \in \partial\Omega \setminus \{\xi\}$.

Hence, $\overline{w}(x)$ is a barrier function at $\xi \in \partial\Omega$.