Karlsruhe Institute of Technology
Institute for Analysis
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WS 2019/2020
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Classical Methods for Partial Differential Equations

Exam sheet WS 2019/2020

First name:	
Family name:	
Matriculation number:	

- The exam sheet consists of eight problems. A maximum of 29 points (plus 4 extra bonus points in Problem 4) can be achieved. 10 points are required to pass the exam.
- The exam will take 150 minutes.
- Write your name and matriculation number on each of the sheets.
- Do not use pencil or red ink.
- You can write your solutions in English or German.

To be completed by the corrector:

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	Σ	grade

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Problem 1 (2 points)

Solve the following initial value problem for the three dimensional homogeneous wave equation:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2}(x,t) - \Delta u(x,t) = 0, & \text{for} \quad (x,t) \in \mathbb{R}^3 \times [0,\infty), \\ u(x,0) = x_1 x_2 + x_2 x_3 + x_1 x_3, & \text{for} \quad x \in \mathbb{R}^3, \\ \frac{\partial u}{\partial t}(x,0) = 2x_1 + x_2^2 - x_3^2, & \text{for} \quad x \in \mathbb{R}^3. \end{cases}$$

Hint: The functions $x_i x_j$, $x_i^2 - x_j^2$ $(i \neq j)$ are harmonic.

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Problem 2

1. (3 points) Derive an integral representation for the solution of the following initial value problem for the 1D wave equation. You may use Duhamel's principle and the Cauchy problem for the function V arising there, but not the fact that the desired integral representation was derived in the exercises.

$$\begin{cases} \frac{\partial^2 u}{\partial t^2}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) = w(x,t), & \text{for} \quad (x,t) \in \mathbb{R} \times [0,\infty), \\ u(x,0) = u_0(x), & \text{for} \quad x \in \mathbb{R}, \\ \frac{\partial u}{\partial t}(x,0) = u_1(x), & \text{for} \quad x \in \mathbb{R}, \end{cases}$$

   where $w \in C^2(\mathbb{R} \times [0,\infty))$, $u_0 \in C^2(\mathbb{R})$ and $u_1 \in C^1(\mathbb{R})$ are given.

2. (1 point) Let $w(x,t) = x$, $u_0(x) = x$, $u_1(x) = \cos x$. Find the explicit solution of the above problem.

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Problem 3 (5 points)

Let $c > 0$ be a constant. Let $u_0(x) = \sum_{k=1}^{N} a_k \sin(kx)$, with $N \in \mathbb{N}$ and $a_1, ..., a_N \in \mathbb{R}$. Solve, by separation of variables, the problem

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) - c\frac{\partial^2 u}{\partial x^2}(x,t) = 0, & \text{for} \quad (x,t) \in (0,\pi) \times (0,\infty), \\ u(x,0) = u_0(x), & \text{for} \quad x \in [0,\pi], \\ u(0,t) = u(\pi,t) = 0, & \text{for} \quad t \in (0,\infty). \end{cases}$$

What happens with the solution when $t \to \infty$? Give a physical interpretation of this result.

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Problem 4

Let $\Omega = \{x \in \mathbb{R}^n : |x| < R\}$, and let $\varphi \in C(\partial\Omega)$. Recall the Poisson formula

$$u(x) = \frac{R^2 - |x|^2}{R\omega_n} \int_{\partial\Omega} \frac{\varphi(\xi)}{|x-\xi|^n} d\sigma(\xi), \quad x \in \Omega,$$

where ω_n is the surface area of the unit sphere in $\mathbb{R}^n$.

1. (4 points) Show that $\Delta u = 0$ in Ω and $u \in C^\infty(\Omega)$.
2. (4 extra bonus points) Show that for each $x_0 \in \partial\Omega$, $\lim_{\substack{x \to x_0 \\ x \in \Omega}} u(x) = \varphi(x_0)$.

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Problem 5 (2 points)

Let u be harmonic on $\mathbb{R}^n$. Show that the function

$$v(x) = u(Tx) \quad (x \in \mathbb{R}^n)$$

is harmonic on $\mathbb{R}^n$, where T is an orthogonal matrix.

Problem 6

Let $\Omega \subset \mathbb{R}^n$ be a domain.

1. (3 points) Let $v : \Omega \to \mathbb{R}$ be subharmonic. Show that

$$v(x) \leqslant \frac{1}{|B(x,r)|} \int_{B(x,r)} v(y)\, dy$$

for all $x \in \Omega$ and $r > 0$ sufficiently small, where $B(x,r)$ denotes the ball with radius r centered at x, and $|B(x,r)|$ its Lebesgue measure.

2. (2 points) Let $u : \Omega \to \mathbb{R}$ be harmonic, and let $\phi \in C^2(\mathbb{R})$ be convex. Show that $\phi(u)$ and $|\operatorname{grad} u|^2$ are subharmonic.

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Problem 7 (2 points)

Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be continuous and bounded, and

$$\varphi(x) = \begin{cases} 0, & \text{for} \quad x \in \mathbb{Q}^n, \\ \psi(x), & \text{for} \quad x \in \mathbb{R}^n \setminus \mathbb{Q}^n. \end{cases}$$

Determine

$$\lim_{\substack{x \to x_0 \\ t \to 0^+}} \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-\xi|^2}{4t}} \varphi(\xi)\, d\xi$$

for $x_0 \in \mathbb{R}^n$.

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Problem 8 (5 points)

Determine the type of the following second order partial differential equation, reduce it to its normal form and obtain its general solution:

$$x^2 \frac{\partial^2 u}{\partial x^2}(x, y) = y^2 \frac{\partial^2 u}{\partial y^2}(x, y), \quad \text{for} \quad (x, y) \in (0, \infty)^2.$$