Karlsruhe Institute of Technology
Institute for Analysis
Prof. Dr. Michael Plum
M.Sc. Zihui He

WS 2019/2020
spring holiday

Classical Methods for Partial Differential Equations

Holiday exercise sheet

Exercise 53

Prove the theorem 7.1 in the stated generality.

Hint: Use a weight function $e^{K(|x-h(y)|+|y-g(x)|)}$.

Exercise 54

Let $g \in \mathcal{C}^1([a,b])$ be such that $g(a) = 0$ and $g'(x) > 0$ $(x \in [a,b])$. Moreover let $\Omega = \{(x,y) \in \mathbb{R}^2 \colon a < x < b, 0 < y < g(x)\}$. Consider the initial value problem

$$\begin{cases} \frac{\partial^2 u}{\partial x \partial y}(x,y) &= d\Big(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\Big) \quad ((x,y) \in \Omega)\,, \\ u(x,0) &= \varphi(x) \quad (x \in [a,b])\,, \\ u(x,g(x)) &= \psi(x) \quad (x \in [a,b])\,, \end{cases}$$

where $d \colon \overline{\Omega} \times \overline{\Omega} \times \mathbb{R}^3 \to \mathbb{R}$ is continuous and satisfies the Lipschitz condition of the form

$$|d(x,y,u,p,q) - d(x,y,\widetilde{u},\widetilde{p},\widetilde{q})| \leqslant L(|u-\widetilde{u}| + |p-\widetilde{p}| + |q-\widetilde{q}|)\,,$$

for some constant $L > 0$. Assume that $\varphi, \psi \in \mathcal{C}^1([a,b])$ and $\varphi(a) = \psi(a)$.

Are the curves

$$\Gamma_1 = \{(x,0) \in \mathbb{R}^2 \colon x \in [a,b]\}\,, \Gamma_2 = \{(x,g(x)) \in \mathbb{R}^2 \colon x \in [a,b]\}\,,$$

non-characteristic?

Show that under these assumptions the above problem has an unique solution u satisfying $u \in \mathcal{C}(\overline{\Omega}) \cap \mathcal{C}^1(\overline{\Omega})$ and $\frac{\partial^2 u}{\partial x \partial y} \in \mathcal{C}(\Omega)$.

Exercise 55

Consider the differential equation

$$a\frac{\partial^2 u}{\partial x^2} + 2b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2}{\partial y^2} - \frac{\partial^2 u}{\partial z^2} = 0,$$

with $a,b,c \in \mathbb{R}$, $a > 0$, $c > 0$, $ac > b^2$. Let

$$M = \Big\{(x,y,z) \in \mathbb{R}^3 \colon \varphi(x,y,z) = (z-1)^2 - x^2 - y^2 = 0\Big\} \setminus (0,0,1).$$

1. Determine the characteristic directions.
2. Let $b = 0$. Which points of M are characteristic?

3. For $a = 2$, $b = 0$, $c = 1$ transform the differential equation according to the change of coordinates $\xi = x$, $\eta = y$, $t = \varphi(x, y, z)$.