Exercise 53

Prove the theorem 7.1 in the stated generality.

**Hint:** Use a weight function $e^{K|x-h(y)|+|y-g(x)|}$.

Exercise 54

Let $g \in C^1([a,b])$ be such that $g(a) = 0$ and $g'(x) > 0 \ (x \in [a,b])$. Moreover let $\Omega = \{(x,y) \in \mathbb{R}^2: a < x < b, 0 < y < g(x)\}$. Consider the initial value problem

\[
\begin{cases}
\frac{\partial^2 u}{\partial x \partial y}(x,y) = d(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}) \quad ((x,y) \in \Omega), \\
u(x,0) = \varphi(x) \quad (x \in [a,b]), \\
u(x,g(x)) = \psi(x) \quad (x \in [a,b]),
\end{cases}
\]

where $d: \bar{\Omega} \times \bar{\Omega} \times \mathbb{R}^3 \rightarrow \mathbb{R}$ is continuous and satisfies the Lipschitz condition of the form

$$|d(x,y,u,p,q) - d(x,y,\bar{u},\bar{p},\bar{q})| \leq L(|u - \bar{u}| + |p - \bar{p}| + |q - \bar{q}|),$$

for some constant $L > 0$. Assume that $\varphi, \psi \in C^1([a,b])$ and $\varphi(a) = \psi(a)$.

Are the curves 

$$\Gamma_1 = \{(x,0) \in \mathbb{R}^2: x \in [a,b]\}, \Gamma_2 = \{(x,g(x)) \in \mathbb{R}^2: x \in [a,b]\},$$

non-characteristic?

Show that under these assumptions the above problem has an unique solution $u$ satisfying $u \in C(\bar{\Omega}) \cap C^1(\bar{\Omega})$ and $\frac{\partial^2 u}{\partial x \partial y} \in C(\Omega)$.

Exercise 55

Consider the differential equation

$$a \frac{\partial^2 u}{\partial x^2} + 2b \frac{\partial^2 u}{\partial x \partial y} + c \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial z^2} = 0,$$

with $a, b, c \in \mathbb{R}$, $a > 0$, $c > 0$, $ac > b^2$. Let

$$M = \{(x, y, z) \in \mathbb{R}^3: \varphi(x,y,z) = (z-1)^2 - x^2 - y^2 = 0\} \setminus (0,0,1).$$

1. Determine the characteristic directions.

2. Let $b = 0$. Which points of $M$ are characteristic?
3. For $a = 2$, $b = 0$, $c = 1$ transform the differential equation according to the change of coordinates $\xi = x$, $\eta = y$, $t = \varphi(x, y, z)$. 