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Classical Methods for Partial Differential Equations
Exercise sheet 1

Exercise 1

Determine the order of the following PDE 's. Check if they are linear or non-linear. Find a F as in the lecture with $F\left(x, u_{x_i}, \ldots, u_{x_i x_j}, \ldots, etc.\right) = 0$ if u solves the PDE.

1. Schrödinger $i\hbar u_t + \frac{\hbar^2}{2m}\Delta u = V(x,y,z)\,u$, where the constants $\hbar$, m and the function $V : \mathbb{R}^3 \to \mathbb{R}$ are given. This equation models the wave function of an electron moving in the potential V. $\hbar$ is the normalized Planck's constant and m is the electron's mass.
2. biharmonic $\Delta^2 u = f(x,y)$, where $f : \mathbb{R}^2 \to \mathbb{R}$ is given and $\Delta^2 v = \Delta(\Delta v) = \sum_{i,j=1}^{2} \frac{\partial^2}{\partial x_i^2}\frac{\partial^2}{\partial x_j^2} v$ $= \frac{\partial^4 v}{\partial x^4} + 2\frac{\partial^4 v}{\partial x^2 \partial y^2} + \frac{\partial^4 v}{\partial y^4}$ $\left(v : \mathbb{R}^2 \to \mathbb{R}\right)$. This equation models deflection of a thin plate, where f is the load.
3. eikonal $u_x^2 + u_y^2 = n(x,y)^2$, where $n : \mathbb{R}^2 \to \mathbb{R}$ is given. This equation models the wave fronts in two dimensional geometric optics. The function n describes the refractive index of the material at the point (x,y).
4. Navier-Stokes $\rho\vec{u}_t + \rho(\vec{u}\cdot\nabla)\vec{u} = \mu\Delta\vec{u} - \nabla p, \nabla\cdot\vec{u} = 0$, where μ, ρ are given constants, $\vec{u}\cdot\nabla = \sum_{i=1}^{3} u_i \frac{\partial}{\partial x_i}$ and $\vec{u}$, p are the unknowns. This system models the incompressible fluid flow where ρ is the mass density, μ is the viscosity coefficient, $\vec{u}$ is the flow velocity and p is the hydrostatic pressure.
5. Korteweg-de Vries $u_t + 6uu_x + u_{xxx} = 0$. This equation models waves on shallow water surfaces.
6. Black-Scholes $u_t + \frac{1}{2}\sigma^2 x^2 u_{xx} + rxu_x - ru = 0$, where σ, r are given constants. This equation describes the price of an option. It can be derived from Black-Scholes model.

Exercise 2

Let $v, w : \mathbb{R}^3 \to \mathbb{R}^3$ be continuously differentiable and let u be twice continuously differentiable. Prove the following identities:

1. $\operatorname{div}\operatorname{grad} u = \Delta u$ $\left(u : \mathbb{R}^3 \to \mathbb{R}\right)$,
2. $\operatorname{div}\operatorname{curl} u = 0$ $\left(u : \mathbb{R}^3 \to \mathbb{R}^3\right)$,
3. $\operatorname{curl}\operatorname{grad} u = 0$ $\left(u : \mathbb{R}^3 \to \mathbb{R}\right)$,
4. $\operatorname{curl}\operatorname{curl} u = \operatorname{grad}\operatorname{div} u - \Delta u$ $\left(u : \mathbb{R}^3 \to \mathbb{R}^3\right)$, where $\Delta u = \begin{pmatrix} \Delta u_1 \\ \Delta u_2 \\ \Delta u_3 \end{pmatrix}$, for $u : \mathbb{R}^3 \to \mathbb{R}^3$,
5. $\operatorname{div}(v \times w) = w \cdot \operatorname{curl} v - v \cdot \operatorname{curl} w$.

Note: *curl* = *rot*.

Exercise 3

Assume that the fields $E,\ B : \mathbb{R}^3 \times [0,\infty) \to \mathbb{R}^3$ are twice continuously differentiable and solve the equations

$$\operatorname{curl} E = -\frac{\partial B}{\partial t},\ \operatorname{curl} B = \frac{\partial E}{\partial t}.$$

Show that $\operatorname{div} E$ and $\operatorname{div} B$ do not depend on t.

Conclude that in an electrodynamical system, when all four Maxwell's equations are satisfied, vanishing of the current density j implies that the charge density ϱ does not depend on t ("Time dependent charge density implies a nonzero current").

Exercise 4

Assume that the fields $E, B : \mathbb{R}^3 \times [0,\infty) \to \mathbb{R}^3$ are twice continuously differentiable and solve the Maxwell's equations

$$\operatorname{curl} E = -\frac{\partial B}{\partial t},\ \operatorname{curl} B = j + \frac{\partial E}{\partial t},\ \operatorname{div} B = 0,\ \operatorname{div} E = \varrho,$$

with charge density ϱ and current density j. Prove that the conservation of charge holds, i.e. $\frac{\partial \varrho}{\partial t} + \operatorname{div} j = 0$.