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# Classical Methods for Partial Differential Equations

Exercise sheet 10

### Exercise 34

Let $u_1, \ldots, u_n$ be solutions of the one dimensional heat equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}\,(x \in \mathbb{R}, t > 0)$. Show that the function

$$u(x,t) = u(x_1, \ldots, x_n, t) = \prod_{i=1}^{n} u_i(x_i, t) \quad (x \in \mathbb{R}^n, t > 0)\,,$$

is a solution of the $n-$dimensional heat equation $\frac{\partial u}{\partial t} = \triangle u$.

### Exercise 35

Find the solution of the Cauchy problem for the following generalised heat equation

$$\begin{cases} \frac{\partial u}{\partial t} - \triangle u - V \cdot \nabla u + g(t)\, u &= 0 \qquad (x \in \mathbb{R}^n, t \in (0,T))\,, \\ u(x,0) &= \varphi(x) \quad (x \in \mathbb{R}^n)\,, \end{cases}$$

where $V \in \mathbb{R}^n$, $g \in \mathcal{C}([0,\infty))$. The function $\varphi$ and $T > 0$ are as in Theorem 4.1 in the lecture.

*Hint:* By setting $v(x,t) = u(x - tV, t)$ and $w(x,t) = \psi(t)\, v(x,t)$ reduce the above equation to the classical heat equation.

### Exercise 36

Let $\varphi \in \mathcal{C}(\mathbb{R}^n)$ have a compact support, i.e. $\operatorname{supp} \varphi = \overline{\{x \in \mathbb{R}^n \colon \varphi(x) \neq 0\}}$ is a compact set. Moreover let $K(x,\xi,t)$ denote the heat kernel, and let $u \in \mathcal{C}(\mathbb{R}^n \times [0,\infty)) \cap \mathcal{C}^\infty(\mathbb{R}^n \times (0,\infty))$ be defined as

$$u(x,t) = \int_{\mathbb{R}^n} K(x,\xi,t)\, \varphi(\xi)\ \mathrm{d}\xi.$$

Thus $u$ a solution of the Cauchy Problem

$$\frac{\partial u}{\partial t} = \triangle u \quad (x \in \mathbb{R}^n, t > 0)\,,\ u(x,0) = \varphi(x) \quad (x \in \mathbb{R}^n)\,,$$

(cf. Theorem 4.1). Show that $\lim_{t\to\infty} u(x,t) = 0$ uniformly in $x \in \mathbb{R}^n$.

### Exercise 37

Assume that $\varphi \in \mathcal{C}^k(\mathbb{R}^n)$ is such that $\varphi$ and all of its derivatives up to order $k$

are bounded on $\mathbb{R}^n$. Let $u$ be as in exercise 36. Define $\mathrm{D}_x^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n}}$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$, $|\alpha| = \sum_{i=1}^n \alpha_i$.

1. Prove that, for all $\alpha \in \mathbb{N}_0^n$ such that $|\alpha| \leqslant k$,

$$(\mathrm{D}_x^\alpha u)(x,t) = \int_{\mathbb{R}^n} K(x,\xi,t)\big(\mathrm{D}_\xi^\alpha \varphi\big)(\xi)\ \mathrm{d}\xi \quad (x \in \mathbb{R}^n, t > 0)\,.$$

2. Show that $\mathrm{D}_x^\alpha u \in \mathcal{C}(\mathbb{R}^n \times [0,\infty))$ for all $\alpha \in \mathbb{N}_0^n$ such that $|\alpha| \leqslant k$.
3. What can be said about the derivatives with respect to $t$ and about mixed (with respect to $t$ and $x$) derivatives?

**Christmas special exercise**

Calculate the area of the Sierpiński Christmas tree (cf. Figure 1).

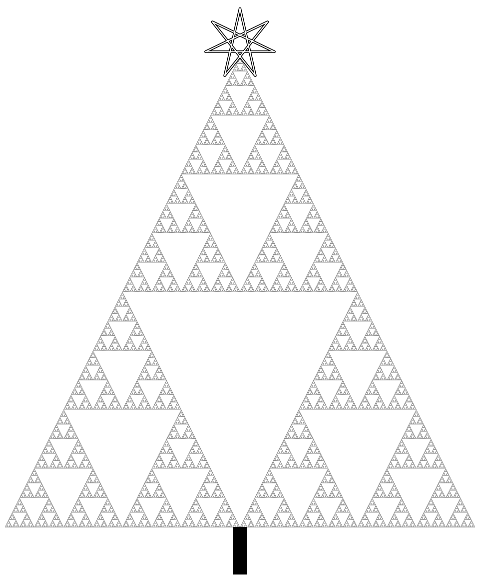

Figure 1: Sierpiński Christmas tree.