Karlsruhe Institute of Technology (KIT)
Institute for Analysis
Prof. Dr. Michael Plum
M.Sc. Zihui He

WS 2019/2020
15.01.2020

Classical Methods for Partial Differential Equations

Exercise sheet 11

Exercise 38

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded Lipschitz domain, $Q = \Omega \times (0,T)$ and $u \in \mathcal{C}^2\left(\overline{Q}\right)$ satisfy

$$\frac{\partial u}{\partial t} - \triangle u \leqslant 0 \text{ in } Q.$$

Give an alternative proof of the maximum principle

$$\max_{\overline{Q}} u = \max_{\partial' Q} u,$$

using the *energy method*: Let $M = \max_{\partial' Q} u$, $\psi(y) = \max\{y - M, 0\}^4$ $(y \in \mathbb{R})$. Prove the following statements

1. $\psi(u) = \psi \circ u \in \mathcal{C}^2\left(\overline{Q}\right)$,
2. $\frac{\partial \psi(u)}{\partial t} - \triangle(\psi(u)) \leqslant 0$ in Q,
3. the mapping $t \longmapsto \int_\Omega \psi(u(x,t)) \, \mathrm{d}x$ is monotonically decreasing,
4. $u(x,t) \leqslant M$ $((x,t) \in Q)$.

Can the assumption $u \in \mathcal{C}^2\left(\overline{Q}\right)$ be weakened?

Exercise 39

Consider the inhomogeneous heat equation

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) - \triangle u(x,t) &= w(x,t) \quad (x \in \mathbb{R}^n, t \in (0,T)), \\ u(x,0) &= \varphi(x) \quad (x \in \mathbb{R}^n), \end{cases}$$

where $\varphi \in \mathcal{C}(\mathbb{R}^n)$, $T > 0$ fulfil the conditions from the Theorem 4.1 and $w \in \mathcal{C}(\mathbb{R}^n \times (0,T))$.

1. Apply Duhamel's principle to the inhomogeneous heat equation. Under which assumptions on the function w does this method actually give a solution?
2. Determine a solution of the inhomogeneous heat equation in the case of $n = 1$ for $w(x,t) = x^2 t^2$, $\varphi(x) = x^2$.

Exercise 40

Let $\phi(t) = e^{-\frac{1}{t^2}} \mathbb{1}_{\{t>0\}}$ and define $u(x,t) = \begin{cases} \sum_{k=0}^{\infty} \phi^{(k)}(t) \frac{x^{2k}}{(2k)!} & , t > 0, \\ 0, & t = 0. \end{cases}$

Show that $u \in C(\mathbb{R} \times [0,\infty)) \cap C^{\infty}(\mathbb{R} \times (0,\infty))$ and that u satisfies

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad (x \in \mathbb{R}, t > 0)\,, \qquad u(x,0) = 0 \quad (x \in \mathbb{R}).$$

Hint: Make use of the Cauchy formula for calculating an upper bound for $|\frac{\partial^k \phi}{\partial t^k}|$:

$$\phi^{(k)}(t) = \frac{k!}{2\pi i} \int_{\Gamma} \frac{\phi(z)}{(z-t)^{k+1}}\, dz\ ,$$

where Γ is a simple closed path in the z-plane such that the point $(t,0)$ is inside Γ and the origin is outside.

Exercise 41

Solve the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad (0 < x < \pi,\ t > 0)$$

with the following boundary conditions by separation of variables:

1. $u(0,t) = u(\pi,t) = 0$;
2. $\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(\pi,t) = 0$.