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Classical Methods for Partial Differential Equations

Exercise sheet 12

Exercise 42

Let $g \in \mathcal{C}^1([a,b])$ be such that $g'(x) < 0\,(x \in [a,b])$. Consider the curve

$$\Gamma = \{(x, g(x)) : x \in (a,b)\}\,,$$

and the initial value problem

$$\begin{cases} \frac{\partial^2 u}{\partial x \partial y}(x,y) &=& r(x,y) & ((x,y) \in (a,b) \times (g(b)\,, g(a)))\,, \\ u(x,g(x)) &=& U(x) & (x \in (a,b))\,, \\ \frac{\partial u}{\partial x}(x,g(x)) &=& U_1(x) & (x \in (a,b))\,, \\ \frac{\partial u}{\partial y}(x,g(x)) &=& U_2(x) & (x \in (a,b))\,, \end{cases}$$

where $r \in \mathcal{C}\big(\mathbb{R}^2\big)$, $U \in \mathcal{C}^1([a,b])$, $U_1, U_2 \in \mathcal{C}([a,b])$ and $U'(x) = U_1(x) + g'(x)\,U_2(x)$ $(x \in [a,b])$. Show that the above initial value problem has an unique solution and find its integral representation.

Exercise 43

Let $\Omega \subseteq \mathbb{R}^2$ be a domain. Consider the following differential equation

$$a\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)\frac{\partial^2 u}{\partial x^2} + 2b\left(x,y,u,\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\right)\frac{\partial^2 u}{\partial x \partial y} + c\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)\frac{\partial^2 u}{\partial y^2} = d\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right) \quad ((x,y) \in \Omega)$$

with the following initial values, for all $t \in (t_0, t_1)$

$$u(x(t)\,, y(t)) = U(t)\,,\ \frac{\partial u}{\partial x}(x(t)\,, y(t)) = U_1(t)\,,\ \frac{\partial u}{\partial y}(x(t)\,, y(t)) = U_2(t)\,,$$

where $a, b, c, d \in \mathcal{C}^k\big(\Omega \times \mathbb{R}^3\big)$ and the curve $\Gamma = \{(x(t)\,, y(t)) : t \in (t_0, t_1)\} \subseteq \Omega$ is non-characteristic, with x, y in $\mathcal{C}^1((t_0, t_1))$ Let $u \in \mathcal{C}^k(\Omega)$ be a solution of the above initial value problem. Show that the derivatives of u up to order k on Γ can be determined by the data U, U_1, U_2, x, y and a, b, c, d, and their derivatives (without knowing the solution u explicitly).

Exercise 44

Determine the type and find all characteristics of the following differential equations

1. $y\frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) = 0$ (Tricomi equation),
2. $\frac{\partial^2 u}{\partial x^2}(x,y) - \left(1+y^2\right)\frac{\partial^2 u}{\partial y^2}(x,y) = 0$, *Hint:* $\int \frac{1}{\sqrt{1+x^2}}\,\mathrm{d}x = \mathrm{arsinh} x$.

Exercise 45

Consider the minimal surface equation

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0$$

on the annulus

$$\Omega = \left\{(x,y) \in \mathbb{R}^2 \colon r_1 < \sqrt{x^2+y^2} < r_2\right\},\ 0 < r_1 < r_2.$$

Note that the set Ω is not simply connected. Determine all radially symmetric solutions (i.e. depending only on $r = |x|$) of this equation. For which $\varphi_1, \varphi_2 \in \mathbb{R}$, is the corresponding boundary value problem

$$u = \varphi_1 \text{ on } \Gamma_1 = \left\{(x,y) \in \mathbb{R}^2 \colon \sqrt{x^2+y^2} = r_1\right\},$$
$$u = \varphi_2 \text{ on } \Gamma_2 = \left\{(x,y) \in \mathbb{R}^2 \colon \sqrt{x^2+y^2} = r_2\right\},$$

solvable?