Exercise 42

Let $g \in C^1([a, b])$ be such that $g'(x) < 0 \ (x \in [a, b])$. Consider the curve

$$\Gamma = \{(x, g(x)) : x \in (a, b)\},$$

and the initial value problem

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2}(x, y) &= r(x, y) \quad ((x, y) \in (a, b) \times (g(b), g(a))), \\
u(x, g(x)) &= U(x) \quad (x \in (a, b)), \\
\frac{\partial u}{\partial x}(x, g(x)) &= U_1(x) \quad (x \in (a, b)), \\
\frac{\partial u}{\partial y}(x, g(x)) &= U_2(x) \quad (x \in (a, b)),
\end{aligned}$$

where $r \in C(\mathbb{R}^2)$, $U \in C^1([a, b])$, $U_1, U_2 \in C([a, b])$ and $U'(x) = U_1(x) + g'(x) U_2(x)$ $(x \in [a, b])$. Show that the above initial value problem has an unique solution and find its integral representation.

Exercise 43

Let $\Omega \subseteq \mathbb{R}^2$ be a domain. Consider the following differential equation

$$a \left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) \frac{\partial^2 u}{\partial x^2} + 2b \left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) \frac{\partial^2 u}{\partial x \partial y} + c \left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) \frac{\partial^2 u}{\partial y^2}$$

$$= d \left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) \quad ((x, y) \in \Omega)$$

with the following initial values, for all $t \in (t_0, t_1)$

$$u(x(t), y(t)) = U(t), \quad \frac{\partial u}{\partial x}(x(t), y(t)) = U_1(t), \quad \frac{\partial u}{\partial y}(x(t), y(t)) = U_2(t),$$

where $a, b, c, d \in C^k(\Omega \times \mathbb{R}^3)$ and the curve $\Gamma = \{(x(t), y(t)) : t \in (t_0, t_1)\} \subseteq \Omega$ is non-characteristic, with $x, y$ in $C^1((t_0, t_1))$ Let $u \in C^k(\Omega)$ be a solution of the above initial value problem. Show that the derivatives of $u$ up to order $k$ on $\Gamma$ can be determined by the data $U, U_1, U_2, x, y$ and $a, b, c, d$, and their derivatives (without knowing the solution $u$ explicitly).

Exercise 44

Determine the type and find all characteristics of the following differential equations
1. \( y \frac{\partial^2 u}{\partial x^2}(x, y) + \frac{\partial^2 u}{\partial y^2}(x, y) = 0 \) (Tricomi equation),

2. \( \frac{\partial^2 u}{\partial x^2}(x, y) - (1 + y^2) \frac{\partial^2 u}{\partial y^2}(x, y) = 0 \), Hint: \( \int \frac{1}{\sqrt{1+x^2}} \, dx = \text{arsinh} x \).

**Exercise 45**

Consider the minimal surface equation

\[
\text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = 0
\]

on the annulus

\[
\Omega = \left\{ (x, y) \in \mathbb{R}^2 : r_1 < \sqrt{x^2 + y^2} < r_2 \right\}, \quad 0 < r_1 < r_2.
\]

Note that the set \( \Omega \) is not simply connected. Determine all radially symmetric solutions (i.e. depending only on \( r = |x| \)) of this equation. For which \( \varphi_1, \varphi_2 \in \mathbb{R} \), is the corresponding boundary value problem

\[
u = \varphi_1 \text{ on } \Gamma_1 = \left\{ (x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} = r_1 \right\},
\]

\[
u = \varphi_2 \text{ on } \Gamma_2 = \left\{ (x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} = r_2 \right\},
\]

solvable?