Exercise 46

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded and simply connected Lipschitz domain and let $r \in C(\overline{\Omega}, \mathbb{R})$. Analogously as in the lecture, derive the Euler-Lagrange equations for the following variational problems:

1. minimize $J[u] = \int_{\Omega} \left( \frac{1}{2} \| \nabla u \|^2 - ru \right) \, dx$ on the set $\{ u \in C^2(\overline{\Omega}) : u = 0 \text{ on } \partial \Omega \}$,

2. minimize $J[u] = \int_{\Omega} \left( \frac{1}{2} \| \nabla u \|^2 - ru \right) \, dx$ on the set $C^2(\overline{\Omega})$.

Exercise 47

Show that the functional

$$J[u] := \int_{0}^{1} \left( \frac{1}{2} x^2 (u'(x))^2 - u(x) \right) \, dx$$

admits no minimum in $\{ u \in C^2([0,1]) \mid u(0) = u(1) = 0 \}$.

Exercise 48

Let $\Omega \subseteq \mathbb{R}^n$ be a domain and $\tilde{a} \in C^1(\Omega \times \Omega, \mathbb{R})$, $\gamma \in C(\Omega \times \Omega \times \mathbb{R}, \mathbb{R})$ and $\sigma \in \{-1, 1\}$. Transform the differential equation

$$\frac{\partial^2 u}{\partial x^2} = \tilde{a}(x, y) \frac{\partial u}{\partial x} + \sigma \frac{\partial u}{\partial y} + \gamma(x, y, u),$$

into the normal form

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial y} + \tilde{\gamma}(x, y, v).$$

Hint: Use a transformation of the form $u = vw$, where $w$ is a positive, smooth function.

Exercise 49 (Legendre Transformation)

Let $u \in C^2(\mathbb{R}^2)$ be a solution of

$$a \left( \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right) \frac{\partial^2 u}{\partial x^2} + 2b \left( \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right) \frac{\partial^2 u}{\partial x \partial y} + c \left( \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right) \frac{\partial^2 u}{\partial y^2} = 0$$
with
\[
\frac{\partial^2 u \, \partial^2 u}{\partial x^2 \, \partial y^2} \neq \left( \frac{\partial^2 u}{\partial x \partial y} \right)^2.
\]

Introduce the new independent variables \( \xi, \mu \) and a new function \( \phi(\xi, \mu) \) by setting
\[
\xi = \frac{\partial u}{\partial x}(x, y) \quad \mu = \frac{\partial u}{\partial y}(x, y) \quad \phi(x, y) = x \frac{\partial u}{\partial x}(x, y) + y \frac{\partial u}{\partial y}(x, y) - u(x, y).
\]

Show that \( \phi \) satisfies the (linear !) equations
\[
\frac{\partial \phi}{\partial \xi}(\xi, \mu) = x, \quad \frac{\partial \phi}{\partial \mu}(\xi, \mu) = y,
\]
\[
a(\xi, \mu) \frac{\partial^2 \phi}{\partial \mu^2}(\xi, \mu) - 2b(\xi, \mu) \frac{\partial^2 \phi}{\partial \mu \partial \xi}(\xi, \mu) + c(\xi, \mu) \frac{\partial^2 \phi}{\partial \xi^2}(\xi, \mu) = 0.
\]