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Classical Methods for Partial Differential Equations

Exercise sheet 13

Exercise 46

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded and simply connected Lipschitz domain and let $r \in \mathcal{C}\big(\overline{\Omega}, \mathbb{R}\big)$. Analogously as in the lecture, derive the Euler-Lagrange equations for the following variational problems:

1. minimize $J[u] = \int_\Omega \Big(\frac{1}{2}\left|\nabla u\right|^2 - ru\Big)\, \mathrm{d}x$ on the set $\big\{u \in \mathcal{C}^2\big(\overline{\Omega}\big) : u = 0 \text{ on } \partial\Omega\big\}$,
2. minimize $J[u] = \int_\Omega \Big(\frac{1}{2}\left|\nabla u\right|^2 - ru\Big)\, \mathrm{d}x$ on the set $\mathcal{C}^2\big(\overline{\Omega}\big)$.

Exercise 47

Show that the functional

$$J[u] := \int_0^1 \Big(\frac{1}{2}x^2(u'(x))^2 - u(x)\Big)\, dx$$

admits no minimum in $\{u \in C^2([0,1]) \,\Big|\, u(0) = u(1) = 0\}$.

Exercise 48

Let $\Omega \subseteq \mathbb{R}^n$ be a domain and $\widetilde{a} \in \mathcal{C}^1(\Omega \times \Omega, \mathbb{R})$, $\gamma \in \mathcal{C}(\Omega \times \Omega \times \mathbb{R}, \mathbb{R})$ and $\sigma \in \{-1, 1\}$. Transform the differential equation

$$\frac{\partial^2 u}{\partial x^2} = \widetilde{a}(x,y)\,\frac{\partial u}{\partial x} + \sigma\frac{\partial u}{\partial y} + \gamma(x,y,u)\,,$$

into the normal form

$$\frac{\partial^2 v}{\partial x^2} = \sigma\frac{\partial v}{\partial y} + \widetilde{\gamma}(x,y,v)\,.$$

Hint: Use a transformation of the form $u = vw$, where w is a positive, smooth function.

Exercise 49 (Legendre Transformation)

Let $u \in C^2(\mathbb{R}^2)$ be a solution of

$$a\Big(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\Big)\,\frac{\partial^2 u}{\partial x^2} \;+\; 2b\Big(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\Big)\,\frac{\partial^2 u}{\partial x \partial y} \;+\; c\Big(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\Big)\,\frac{\partial^2 u}{\partial y^2} \;=\; 0$$

with

$$\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2} \neq \Big(\frac{\partial^2 u}{\partial x \partial y}\Big)^2.$$

Introduce the new independent variables ξ, μ and a new function $\phi(\xi, \mu)$ by setting

$$\xi = \frac{\partial u}{\partial x}(x, y) \quad \mu = \frac{\partial u}{\partial y}(x, y) \quad \phi(x, y) = x\frac{\partial u}{\partial x}(x, y) + y\frac{\partial u}{\partial y}(x, y) - u(x, y).$$

Show that ϕ satisfies the (linear !) equations

$$\frac{\partial \phi}{\partial \xi}(\xi, \mu) = x\,, \qquad \frac{\partial \phi}{\partial \mu}(\xi, \mu) = y,$$

$$a(\xi, \mu)\frac{\partial^2 \phi}{\partial \mu^2}(\xi, \mu) \; - \; 2b(\xi, \mu)\frac{\partial^2 \phi}{\partial \mu \partial \xi}(\xi, \mu) \; + \; c(\xi, \mu)\frac{\partial^2 \phi}{\partial \xi^2}(\xi, \mu) \; = \; 0.$$