Karlsruhe Institute of Technology — WS 2019/2020
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Classical Methods for Partial Differential Equations

Exercise sheet 2

Exercise 5

Let $\Omega \subseteq \mathbb{R}^3$ be *star shaped*, i.e. there exists a point $x_0 \in \Omega$ such that for every point $x \in \Omega$ and every $t \in [0,1]$, the point $x_0 + t(x - x_0) \in \Omega$. The point x_0 is then called a *centre of the set* Ω. Moreover assume that the set Ω is open.

1. (existence of a scalar potential) Let $v \in \mathcal{C}^1\big(\Omega, \mathbb{R}^3\big)$ be such that $\operatorname{curl} v = 0$. Show that the function $\Phi\colon \Omega \to \mathbb{R}$ defined as

$$\Phi(x) = \int_0^1 (x - x_0) \cdot v(x_0 + t(x - x_0))\ \mathrm{d}t \quad (x \in \Omega)\,,$$

   is continuously differentiable, and $\operatorname{grad} \Phi = v$ in Ω.

2. (existence of a vector potential) Let $w \in \mathcal{C}^1\big(\Omega, \mathbb{R}^3\big)$ be such that $\operatorname{div} w = 0$. Show that the function $A\colon \Omega \to \mathbb{R}^3$ defined as

$$A(x) = \left(\int_0^1 t w(x_0 + t(x - x_0))\ \mathrm{d}t\right) \times (x - x_0) \quad (x \in \Omega)\,,$$

   is continuously differentiable, and $\operatorname{curl} A = w$ in Ω.

Exercise 6

Let $u \in \mathcal{C}^1\big(\mathbb{R}^3, \mathbb{R}^3\big)$. Assume that there exists a constant $c > 0$ such that

$$|u(x)| \leqslant \frac{c}{|x|^3 + 1} \quad \big(x \in \mathbb{R}^3\big)\,,$$

and that $\int_{\mathbb{R}^3} |\operatorname{div} u(x)|\ \mathrm{d}x < \infty$. Show that

$$\int_{\mathbb{R}^3} (\operatorname{div} u)(x)\ \mathrm{d}x = 0.$$

Hint: Integrate over the ball $\mathrm{B}_R(0)$ and take $R \to \infty$.

Exercise 7

Deduce from the wave equations for the potentials Φ and A (Lorentz-gauge) wave equations for the fields E and B.

Exercise 8

Consider the incompressible, stationary Euler equation

$$\rho\big(\overrightarrow{v} \cdot \nabla\big)\, \overrightarrow{v} = -\nabla p,\ \nabla \cdot \overrightarrow{v} = 0,$$

where the mass density $\rho > 0$ of the fluid is assumed to be constant, $\overrightarrow{v} : \mathbb{R}^3 \to \mathbb{R}^3$ is the flow velocity vector field, and $p \colon \mathbb{R}^3 \to \mathbb{R}$ is the hydrostatic pressure. Moreover, let the flow be irrotational, i.e. $\operatorname{curl} \overrightarrow{v} = 0$. Show that

$$\frac{1}{2}\rho\,|\overrightarrow{v}|^2 + p,$$

is constant (Bernoulli's theorem: "the sum of dynamic pressure and hydrostatic pressure is constant"). *Hint:* Show first that $\frac{1}{2}\nabla\Big(|\overrightarrow{v}|^2\Big) = \big(\overrightarrow{v} \cdot \nabla\big)\, \overrightarrow{v} + \overrightarrow{v} \times \operatorname{curl} \overrightarrow{v}$.