

Classical Methods for Partial Differential Equations
Exercise sheet 3

Exercise 9

Let u be a solution of a one dimensional wave equation with initial data $u(x, 0) = u_0(x)$ and $\frac{\partial u}{\partial t}(x, 0) = u_1(x)$ ($x \in \mathbb{R}$). Assume that $u_0 \in \mathcal{C}^2(\mathbb{R})$, $u_1 \in \mathcal{C}^1(\mathbb{R})$ and that there exists an $R > 0$ such that $u_0(x) = u_1(x) = 0$ ($|x| > R$). Show that there exists a $t_0 > 0$ such that

$$\frac{1}{2} \int_{\mathbb{R}} \left(\frac{\partial u}{\partial t}(x, t) \right)^2 dx = \frac{1}{2} \int_{\mathbb{R}} \left(\frac{\partial u}{\partial x} \right)^2 dx \quad (t > t_0),$$

i.e. for all times $t > t_0$ kinetic and potential energies are equal.

Exercise 10

Let u be a solution of the n -dimensional wave equation $\frac{\partial^2 u}{\partial t^2} - \Delta u = 0$ of the form $u(x, t) = v(|x|, t)$, where $v \in \mathcal{C}^2([0, \infty) \times [0, \infty), \mathbb{R})$. Such u is called then a *spherical wave*.

1. Show that the function $v = v(r, t)$ satisfies the following equation

$$\frac{\partial^2 v}{\partial t^2} = \frac{\partial^2 v}{\partial r^2} + \frac{n-1}{r} \frac{\partial v}{\partial r}.$$

2. Derive a differential equation for the function $U(r, t) = v(r, t) r^{\frac{n-1}{2}}$.
3. In case $n = 3$ derive a general formula for spherical waves.
4. In the case $n = 3$ find, for $t < r$, a formula for the spherical wave satisfying the following initial conditions

$$u(r, 0) = u_0(r), \quad \frac{\partial u}{\partial t}(r, 0) = u_1(r) \quad (r \in (0, \infty)),$$

where $u_0, u_1 \in \mathcal{C}^2((0, \infty))$ are given.

5. Assume that $u_i \in \mathcal{C}^2([0, \infty))$ and $u_i'(0) = 0$ ($i = 0, 1$). Find a solution of the above problem for all $r > 0$ and $t \geq 0$. Show that u remains bounded for $r \rightarrow 0$.

Exercise 11

Standing waves are the solutions of the wave equation having the form $u(x, t) = v(x) \sin(\omega t + \varphi)$ where $\omega > 0$, $\varphi \in [0, 2\pi)$ and $v \in \mathcal{C}^2(\mathbb{R}^n)$.

1. Derive a differential equation for the function v .
2. Let $n = 1$. Determine all standing waves. Represent each standing wave as a superposition of two waves, one travelling to the left and one to the right.
3. In case of $n = 3$ find all standing spherical waves, that is all solutions of the wave equation having the form $u(x, t) = v(|x|) \sin(\omega t + \varphi)$, where $v \in \mathcal{C}^2((0, \infty))$.