Karlsruhe Institute of Technology
Institute for Analysis
Prof. Dr. Michael Plum
M.Sc. Zihui He

WS 2019/2020
13.11.2019

Classical Methods for Partial Differential Equations

Exercise sheet 4

Exercise 12

Solve the following initial value problem for the three dimensional homogeneous wave equation

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \triangle u &= 0 & \text{in } \mathbb{R}^3 \times (0,\infty)\,, \\ u(x,0) &= 0 & \big(x \in \mathbb{R}^3\big)\,, \\ \frac{\partial u}{\partial t}(x,0) &= x_1^2 + x_1 x_2 + x_3^2 & \big(x = (x_1,x_2,x_3) \in \mathbb{R}^3\big)\,. \end{cases}$$

In the following exercise we will use the notation

$$\mathrm{D}^\alpha u = \frac{\partial^{|\alpha|}}{\partial \widehat{x}_1^{\alpha_1} \ldots \partial \widehat{x}_n^{\alpha_n}},$$

where $\alpha = (\alpha_1, \ldots \alpha_n) \in \mathbb{N}_0^n$, $|\alpha| = \alpha_1 + \ldots + \alpha_n$ and $\widehat{x} = (\widehat{x}_1, \ldots, \widehat{x}_n) = (x_1, \ldots, x_{n-1}, t)$. Moreover we use the convention that $\mathbb{R}^3 \times [0,\infty) \subseteq \mathbb{R}^4$.

Exercise 13

Let $u \in \mathcal{C}^2\big(\mathbb{R}^3 \times [0,\infty)\big)$ be a solution of the three dimensional homogeneous wave equation and suppose that $U(0) < \infty$, where

$$U(t) = \sum_{|\alpha| \leqslant 2} \int_{\mathbb{R}^3} |\mathrm{D}^\alpha u(x,t)| \,\mathrm{d}x.$$

1. Show that there exists a constant $K > 0$ such that

$$|u(x,t)| \leqslant \frac{K}{t} U(0) \quad \big(x \in \mathbb{R}^3, t \geqslant 1\big)\,.$$

   Hint: Let $S = \big\{y \in \mathbb{R}^3 \colon |y| = 1\big\}$. Write the solution as a integral of the form $\int_S f \cdot \nu \,\mathrm{d}\sigma$ with a suitable function f and apply the divergence theorem.

2. Show that if $\lim_{t\to\infty} \frac{U(t)}{t} = 0$, then $u(x,t) = 0$ $\big((x,t) \in \mathbb{R}^3 \times [0,\infty)\big)$. *Hint:* Apply the above result for the function $v(x,t) = u(x, T-t)$ ($T > 0$ sufficiently large).

Exercise 14

Let $u \in \mathcal{C}^2(\mathbb{R}^n \times [0,\infty))$, $n \leqslant 3$ be a solution of the homogeneous wave equation.

Suppose that $u(x,0) = 0$ and $\frac{\partial u}{\partial t}(x,0) = 0$ for all $x \in \mathbb{R}^n$ such that $|x| > R$, for some constant $R > 0$. Show that the energy integral

$$E(t) = \frac{1}{2} \int_{\mathbb{R}^n} \left(\frac{\partial u}{\partial t}(x,t) \right)^2 + \left| \nabla u(x,t) \right|^2 \, \mathrm{d}x,$$

does not depend on time t.