

**Classical Methods for Partial Differential Equations**  
Exercise sheet 5

**Exercise 15** 1. Applying Duhamel's principle solve the following initial value problem for the one dimensional wave equation:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = w(x, t) & \text{in } \mathbb{R} \times [0, \infty), \\ u(x, 0) = u_0(x) & (x \in \mathbb{R}), \\ \frac{\partial u}{\partial t}(x, 0) = u_1(x) & (x \in \mathbb{R}). \end{cases}$$

2. Let  $w(x, t) = x^2$ ,  $u_0(x) = x$ ,  $u_1(x) = 0$ . Find the explicit solution of the above problem.

**Exercise 16**

Let  $u \in C^4(\mathbb{R} \times [0, \infty))$ ,  $c > 0$ ,  $c \neq 1$  and

$$\left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right) \left( \frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2} \right) u(x, t) = 0.$$

Show that there exist functions  $g_i \in C^4(\mathbb{R})$  ( $i \in \{1, \dots, 4\}$ ) such that

$$u(x, t) = g_1(x + ct) + g_2(x - ct) + g_3(x + t) + g_4(x - t).$$

**Exercise 17**

Let  $\Omega = (0, a) \times (0, b)$  (for some  $a, b > 0$ ),  $u_0 \in C^2(\overline{\Omega})$  and  $\frac{\partial u_0}{\partial \nu} = 0$  on  $\partial\Omega$ . Using separation of variables find the solution of the following boundary value problem for the heat equation:

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u & \text{in } \Omega \times (0, \infty) \\ u(x, 0) = u_0(x) & (x \in \Omega) \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, \infty) \end{cases}$$

What happens with the solution when  $t \rightarrow \infty$ ? Give a physical interpretation of this result.

**Hint:** Use without proof that the Fourier series  $u_0(x_1, x_2) = \sum_{n,m=0}^{\infty} a_{n,m} \cos\left(\frac{n\pi}{a}x_1\right) \cos\left(\frac{m\pi}{b}x_2\right)$  converges uniformly, where

$$a_{n,m} = \frac{c_{n,m}}{ab} \int_{\Omega} u_0(x_1, x_2) \cos\left(\frac{n\pi}{a}x_1\right) \cos\left(\frac{m\pi}{b}x_2\right) dx_1 dx_2 \quad (n, m \in \mathbb{N}_0),$$

and

$$c_{n,m} = \begin{cases} 1 & (n = m = 0), \\ 2 & (n = 0, m \in \mathbb{N}), \\ 4 & (n, m \in \mathbb{N}). \end{cases}$$