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Classical Methods for Partial Differential Equations

Exercise sheet 6

Exercise 18

Let $\Omega = (0,a) \times (0,b)$ for some $a, b > 0$. Using separation of variables find the eigenvalues and the eigenfunctions of the following problem (with periodic boundary conditions):

$$\begin{cases} -\triangle u + \frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = \lambda u & \text{in } \Omega, \\ u(0,y) = u(a,y)\,, \quad \frac{\partial u}{\partial x}(0,y) = \frac{\partial u}{\partial x}(a,y) & (y \in (0,b))\,, \\ u(x,0) = u(x,b)\,, \quad \frac{\partial u}{\partial y}(x,0) = \frac{\partial u}{\partial y}(x,b) & (x \in (0,a))\,. \end{cases}$$

Exercise 19

1. In $\mathbb{R}^2$ the polar coordinates are introduced as follows: $x = r\cos\varphi$, $y = r\sin\varphi$. Show that

$$\triangle = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}.$$

2. In $\mathbb{R}^3$ the spherical coordinates can be introduced as follows:

$$x = r\sin\vartheta\cos\varphi,\ y = r\sin\vartheta\sin\varphi,\ z = r\cos\vartheta.$$

Show that

$$\triangle = \frac{1}{r^2}\left(\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\sin\vartheta\frac{\partial}{\partial\vartheta} + \frac{1}{\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2}\right).$$

3. In $\mathbb{R}^3$ the cylindrical coordinates are introduced as follows: $x = \rho\cos\varphi$, $y = \rho\sin\varphi$, $z = z$. Show that

$$\triangle = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial z^2}.$$

Exercise 20

Let $\Omega \subseteq \mathbb{R}^n$ be an open set with Lipschitz boundary, $c \in \mathcal{C}(\overline{\Omega})$ be such that $c(x) \geqslant 0$ $(x \in \Omega)$.

1. Show that the Dirichlet boundary value problem

$$\begin{cases} -\triangle u + cu = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has only the trivial solution.

2. Show that the Neumann boundary value problem

$$\begin{cases} -\triangle u + cu &= 0 \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial\Omega, \end{cases}$$

has only the trivial solution, if $c(\xi) > 0$ for some $\xi \in \Omega$. Which non-trivial solutions occur in the case $c = 0$?

3. Let $\gamma \colon \partial\Omega \to [0, \infty)$ be a continuous function. What can be said about the solutions of the Robin boundary value problem

$$\begin{cases} -\triangle u + cu &= 0 \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} + \gamma u &= 0 \quad \text{on } \partial\Omega. \end{cases}$$

Exercise 21

Let $\Omega \subseteq \mathbb{R}^n$ be a Lipschitz domain, $r \in \mathcal{C}(\overline{\Omega})$ and $\varphi \in \mathcal{C}(\partial\Omega)$. Show that the equality

$$\int_\Omega r \,\mathrm{d}x = \int_{\partial\Omega} \varphi \,\mathrm{d}\sigma,$$

is a necessary condition for the existence of a solution of the boundary value problem

$$\begin{cases} \triangle u &= r \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} &= \varphi \quad \text{on } \partial\Omega. \end{cases}$$