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WS 2019/2020
11.12.2019

Classical Methods for Partial Differential Equations

Exercise sheet 8

Exercise 25

1. Let $\Omega \subseteq \mathbb{R}^n$ and let $u \in \mathcal{C}(\Omega)$ be given. Moreover assume that $f\colon \mathbb{R} \to \mathbb{R}$ is convex. Show that
   (a) if the function u is harmonic, then $f \circ u$ is subharmonic,
   (b) if, in addition, the function f is monotonically increasing and if u is subharmonic, then $f \circ u$ is subharmonic.
2. For each $\alpha \in \mathbb{R}$ consider a function $x \longmapsto |x|^\alpha$ on $\mathrm{B}_1(0)$, or on $\mathrm{B}_1(0) \setminus \{0\}$ if $\alpha < 0$. Determine for which α the above function is subharmonic, and for which is superharmonic.

Exercise 26

Let $x_0 \in \mathbb{R}^n$, $R > 0$ and $\Omega \subseteq \mathbb{R}^n$ be a domain, such that $\Omega \subseteq \mathrm{B}_R(x_0)$. Moreover, let $q \in \mathcal{C}(\overline{\Omega})$, $\varphi \in \mathcal{C}(\partial\Omega)$ and

$$\overline{K} = \max_{x\in\overline{\Omega}} \{q(x)\,,0\}\,,\ \underline{K} = \min_{x\in\overline{\Omega}} \{q(x)\,,0\}\,,\ \overline{M} = \max_{x\in\partial\Omega} \varphi(x)\,,\ \underline{M} = \min_{x\in\partial\Omega} \varphi(x)\,.$$

Show that if $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ is a solution of the following boundary value problem

$$\begin{cases} -\triangle u &= q \quad \text{in } \Omega, \\ \quad\ \ u &= \varphi \quad \text{on } \partial\Omega, \end{cases}$$

then the following inequality is true

$$\underline{M} + \frac{\underline{K}}{2n}\Big(R^2 - |x-x_0|^2\Big) \leqslant u(x) \leqslant \overline{M} + \frac{\overline{K}}{2n}\Big(R^2 - |x-x_0|^2\Big)\,.$$

Exercise 27

Prove the Harnack inequality: Let $\Omega = \{x \in \mathbb{R}^n\colon\ |x| < R\}$ and $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$. If u is harmonic in Ω and $u(x) \geqslant 0\,(x \in \Omega)$, then

$$\frac{1-\frac{|x|}{R}}{\Big(1+\frac{|x|}{R}\Big)^{n-1}} u(0) \leqslant u(x) \leqslant \frac{1+\frac{|x|}{R}}{\Big(1-\frac{|x|}{R}\Big)^{n-1}} u(0) \quad (x \in \Omega)\,.$$

Exercise 28

Let $\Omega \subseteq \mathbb{R}^n$ be a domain, $n \geqslant 2$, $x_0 \in \Omega$ and $u\colon \Omega \setminus \{x_0\} \to \mathbb{R}$ be a harmonic function. We say that the point x_0 is a *removable singularity of the function* u, if there exists a harmonic function $\widetilde{u}\colon \Omega \to \mathbb{R}$ such that $\widetilde{u}(x) = u(x)$ $(x \in \Omega \setminus \{x_0\})$. Let S denote the usual singularity function defined in the lecture. Show that

1. If $\lim_{x \to x_0} \frac{u(x)}{S(x_0,x)} = 0$, then x_0 is a removable singularity of u.

2. If $\lim_{x \to x_0} \frac{u(x)}{S(x_0,x)} = c$, for some $c \in \mathbb{R}$, then there exists a harmonic function $v\colon \Omega \to \mathbb{R}$ such that $u(x) = cS(x_0, x) + v(x)$ $(x \in \Omega \setminus \{x_0\})$.

Exercise 29

Let $R > 0$, $\Omega = \mathbb{R}^n \setminus \overline{\mathrm{B}_R(0)}$ and $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}\big(\overline{\Omega}\big)$ be a harmonic function such that $\lim_{|x| \to \infty} u(x) = 0$.

1. Show that $\sup_{x \in \Omega} |u(x)| = \max_{x \in \partial\Omega} |u(x)|$.

2. Find an example showing that the above statement is false, if the assumption $\lim_{|x| \to \infty} u(x) = 0$ is not satisfied.