Exercise 30

Let $\Omega \subseteq \mathbb{R}^n$ be a domain and let $N \subseteq \mathbb{R}^n$ be an open neighbourhood of a point $\xi \in \partial \Omega$. A function $w : \overline{\Omega} \cap N \to \mathbb{R}$ is called a local barrier function at the point $\xi$ if the following conditions are satisfied

1. $w \in C(\overline{\Omega} \cap N)$ is subharmonic in $\Omega \cap N$,
2. $w(\xi) = 0$,
3. $w(x) < 0 \ (x \in (\overline{\Omega} \cap N) \setminus \{\xi\})$.

Show that if for a point $\xi \in \partial \Omega$ there exists a local barrier function, then there exists a barrier function at this point.

Hint: Let $B_\varrho(\xi)$ be a ball around $\xi$ with sufficiently small radius $\varrho > 0$ and $m \in \mathbb{R}$ be chosen in a suitable way. Consider a function

$$w(x) = \begin{cases} \max \{m, w(x)\} & (x \in (\overline{\Omega} \cap B_\varrho(\xi)) \setminus \{\xi\}) \\ m & (x \in (\overline{\Omega} \setminus B_\varrho(\xi)) \setminus \{\xi\}) \end{cases}.$$ 

Exercise 31

1. Let $\Omega \subseteq \mathbb{R}^n$ be a domain satisfying the outer ball condition. Show that for each point of the boundary $\partial \Omega$ a barrier function exists.

Hint: Singularity function.

2. A set $K \subseteq \mathbb{R}^2$ is called a cone with an opening angle $\alpha \in (0, 2\pi)$, if $K$ can be transformed by rigid motion (rotations and translations) to a set

$$K_0 = \{(r \cos \varphi, r \sin \varphi) \in \mathbb{R}^2 : 0 < r < \infty, \ 0 < \varphi < \alpha\}.$$ 

Let $\Omega \subseteq \mathbb{R}^2$ be a domain and let $\xi \in \partial \Omega$. Assume that there exists a cone $K$ such that $K \subseteq \mathbb{R}^2 \setminus \Omega$ and $\overline{K} \cap \overline{\Omega} = \{\xi\}$. Show that there exists a local barrier function at the point $\xi$.

Hint: Consider a function $w(r, \varphi) = r^{\frac{\pi}{2\pi - \alpha}} \sin \left(\frac{\pi}{2\pi - \alpha} \varphi\right)$.

Exercise 32

Show that the solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ (provided it exists) of the boundary value problem

$$\begin{cases} -\Delta u + u^3 = 0, & \text{in } \Omega := K_1(0), \\ u = 1, & \text{on } \partial \Omega, \end{cases}$$

is unique and satisfies $0 \leq u(x) \leq 1$, $\forall x \in \overline{\Omega}$. 

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Exercise 33

Let $\Omega \subseteq \mathbb{R}^n$ be a domain, $M > 0$ and $(u_j)_{j \in \mathbb{N}} \subseteq C(\Omega)$ be a sequence of functions which are harmonic in $\Omega$ and satisfy

$$|u_j(x)| \leq M \quad (x \in \Omega, j \in \mathbb{N}).$$

Show that there exists a subsequence $(u_{j_k})_{k \in \mathbb{N}}$ and a function $u \in C(\Omega)$, such that for all open and bounded sets $U \subseteq \Omega$ such that $\overline{U} \subseteq \Omega$

$$u_{j_k} \underset{k \to \infty}{\longrightarrow} u \text{ uniformly in } U.$$

*Hint:* Apply the gradient estimate in theorem 3.13 from the lecture and the Arzela-Ascoli theorem.