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Classical Methods for Partial Differential Equations

Exercise sheet 9

Exercise 30

Let $\Omega \subseteq \mathbb{R}^n$ be a domain and let $N \subseteq \mathbb{R}^n$ be an open neighbourhood of a point $\xi \in \partial\Omega$. A function $w\colon \overline{\Omega} \cap N \to \mathbb{R}$ is called a *local barrier function* at the point ξ if the following conditions are satisfied

1. $w \in \mathcal{C}\big(\overline{\Omega} \cap N\big)$ is subharmonic in $\Omega \cap N$,
2. $w(\xi) = 0$,
3. $w(x) < 0 \ \big(x \in \big(\overline{\Omega} \cap N\big) \setminus \{\xi\}\big)$.

Show that if for a point $\xi \in \partial\Omega$ there exists a local barrier function, then there exists a barrier function at this point.

Hint: Let $\mathrm{B}_\varrho(\xi)$ be a ball around ξ with sufficiently small radius $\varrho > 0$ and $m \in \mathbb{R}$ be chosen in a suitable way. Consider a function

$$\overline{w}(x) = \begin{cases} \max\{m, w(x)\} & \big(x \in \overline{\Omega} \cap \mathrm{B}_\varrho(\xi)\big)\,, \\ m & \big(x \in \overline{\Omega} \setminus \mathrm{B}_\varrho(\xi)\big)\,. \end{cases}$$

Exercise 31

1. Let $\Omega \subseteq \mathbb{R}^n$ be a domain satisfying the outer ball condition. Show that for each point of the boundary $\partial\Omega$ a barrier function exists.

   Hint: Singularity function.

2. A set $K \subseteq \mathbb{R}^2$ is called a *cone with an opening angle* $\alpha \in (0, 2\pi)$, if K can be transformed by rigid motion (rotations and translations) to a set

   $$K_0 = \big\{(r\cos\varphi, r\sin\varphi) \in \mathbb{R}^2 \colon 0 < r < \infty,\ 0 < \varphi < \alpha\big\}\,.$$

   Let $\Omega \subseteq \mathbb{R}^2$ be a domain and let $\xi \in \partial\Omega$. Assume that there exists a cone K such that $K \subseteq \mathbb{R}^2 \setminus \Omega$ and $\overline{K} \cap \overline{\Omega} = \{\xi\}$. Show that there exists a local barrier function at the point ξ.

   Hint: Consider a function $w(r,\varphi) = r^{\frac{\pi}{2\pi-\alpha}} \sin\Big(\frac{\pi}{2\pi-\alpha}\varphi\Big)$.

Exercise 32

Show that the solution $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ (provided it exists) of the boundary value problem

$$\begin{cases} -\Delta u + u^3 = 0, & \text{in } \Omega := K_1(0), \\ u = 1, & \text{on } \partial\Omega, \end{cases}$$

is unique and satisfies $0 \le u(x) \le 1$, $\forall x \in \overline{\Omega}$.

Exercise 33

Let $\Omega \subseteq \mathbb{R}^n$ be a domain, $M > 0$ and $(u_j)_{j\in\mathbb{N}} \subseteq \mathcal{C}(\Omega)$ be a sequence of functions which are harmonic in Ω and satisfy

$$|u_j(x)| \leqslant M \quad (x \in \Omega, j \in \mathbb{N})\,.$$

Show that there exists a subsequence $(u_{j_k})_{k\in\mathbb{N}}$ and a function $u \in \mathcal{C}(\Omega)$, such that for all open and bounded sets $U \subseteq \Omega$ such that $\overline{U} \subseteq \Omega$

$$u_{j_k} \xrightarrow[k\to\infty]{} u \text{ uniformly in } U.$$

Hint: Apply the gradient estimate in theorem `3.13` from the lecture and the Arzela-Ascoli theorem.