**Problem 1:** (2 points.) The solution is given by Kirchhoff's formula:

$$u(x,t) = \frac{\partial}{\partial t}\left(t\left(Mu_0\right)(x,t)\right) + t\left(Mu_1\right)(x,t), \quad \text{for} \quad (x,t) \in \mathbb{R}^3 \times [0,\infty). \quad \text{(1 point.)}$$

Note that $u_0(x) = x_1x_2 + x_2x_3 + x_1x_3$ and $u_1(x) = 2x_1 + x_2^2 - x_3^2$ are harmonic. By the spherical mean value property

$$(Mu_i)(x,t) = u_i(x), \quad (x,t) \in \mathbb{R}^3 \times [0,\infty), \quad i = 0,1.$$

Hence,

$$u(x,t) = (x_1x_2+x_2x_3+x_1x_3)+t\left(2x_1 + x_2^2 - x_3^2\right), \quad \text{for} \quad (x,t) \in \mathbb{R}^3\times[0,\infty). \quad \text{(1 point.)}$$

**Problem 2:** (3+1 points) 1. First consider the problem

21 (0.1)
$$\begin{cases} \frac{\partial^2 u}{\partial t^2}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) = w(x,t), & \text{for} \quad (x,t) \in \mathbb{R} \times [0,\infty), \\ u(x,0) = 0, & \text{for} \quad x \in \mathbb{R}, \\ \frac{\partial u}{\partial t}(x,0) = 0, & \text{for} \quad x \in \mathbb{R}. \end{cases}$$

Take the Ansatz: $v(x,t) = \int_0^t V(x,t-s,s)\,ds$. $v$ solves (0.1) if $V$ solves:

22 (0.2)
$$\begin{cases} \frac{\partial^2 V}{\partial t^2}(x,t,s) - \frac{\partial^2 V}{\partial x^2}(x,t,s) = 0, & \text{for} \quad (x,t,s) \in \mathbb{R} \times [0,\infty) \times [0,\infty), \\ V(x,0,s) = 0, & \text{for} \quad (x,s) \in \mathbb{R} \times [0,\infty), \\ \frac{\partial V}{\partial t}(x,0,s) = w(x,s), & \text{for} \quad (x,s) \in \mathbb{R} \times [0,\infty). \quad \text{(1 point.)} \end{cases}$$

Note that equation (0.2) is a homogeneous wave equation with parameter $s$. The solution of (0.2) is given by d'Alembert's formula

$$V(x,t,s) = \frac{1}{2}\int_{x-t}^{x+t} w(\varrho,s)\,d\varrho, \quad \text{for} \quad (x,t,s) \in \mathbb{R} \times [0,\infty) \times [0,\infty).$$

Therefore the solution of the inhomogeneous problem (0.1) is given by

$$v(x,t) = \frac{1}{2}\int_0^t \int_{x-(t-s)}^{x+t-s} w(\varrho,s)\,d\varrho ds \overset{\xi=t-s}{=} \frac{1}{2}\int_0^t \int_{x-\xi}^{x+\xi} w(\varrho,t-\xi)\,d\varrho d\xi, \quad \text{(1 point.)}$$

hence the solution of the given problem can be written as

$$u(x,t) = \frac{1}{2}\left(u_0(x+t) + u_0(x-t) + \int_{x-t}^{x+t} u_1(s)\,ds\right) + \frac{1}{2}\int_0^t \int_{x-\xi}^{x+\xi} w(\varrho,t-\xi)\,d\varrho d\xi. \quad \text{(1 point.)}$$

2.

$$u(x,t) = x + \frac{1}{2}(\sin(x+t) - sin(x-t)) + \frac{1}{2}xt^2. \quad \text{(1 point.)}$$

**Problem 3:** (5 points.) Making the Ansatz $u(x,t) = v(x)w(t)$, for some $v \in C^2((0,\pi))$ and $w \in C^1((0,\infty))$, we obtain

$$v(x)w'(t) = cv''(x)w(t), \quad \text{for} \quad (x,t) \in (0,\pi) \times (0,\infty).$$

Assuming for the moment that $v(x) \neq 0$, $w(t) \neq 0$ we get, after dividing both sides of the above equality by $v(x)w(t)$

31 (0.3) $$\frac{1}{c}\frac{w'(t)}{w(t)} = \frac{v''(x)}{v(x)} = \lambda, \quad \text{for} \quad (x,t) \in (0,\pi) \times (0,\infty),$$

where $\lambda \in \mathbb{R}$ is a constant. Thus we can split (0.3) into the two equations:

$$w'(t) - \lambda c w(t) = 0$$

with the general solution

$$w(t) = Ce^{\lambda ct}, C \in \mathbb{R},$$

and

32 (0.4) $$v''(x) - \lambda v(x) = 0. \quad \text{(1 point.)}$$

The boundary conditions imply that

$$v(0) = v(\pi) = 0.$$

Determine the eigenvalues and eigenfunctions: $\underline{\lambda = \mu^2 > 0:}$ The general solutions of equation (0.4) is

$$v(x) = C_1 e^{\mu x} + C_2 e^{-\mu x}.$$

By imposing the boundary conditions we find

$$\begin{cases} C_1 + C_2 = 0, \\ e^{\mu\pi} C_1 + e^{-\mu\pi} C_2 = 0, \end{cases}$$

so $C_1 = C_2 = 0$. This gives only the trivial solution. (0.5 point.)
$\underline{\lambda = 0:}$ General solution: $v(x) = C_1 + C_2 x$, imposing the boundary conditions implies that only the trivial solution exists. (0.5 point.)
$\underline{\lambda = -\mu^2 < 0:}$ Now we have the general solution

$$v(x) = C_1 \cos(\mu x) + C_2 \sin(\mu x), \quad v(0) = v(\pi) = 0.$$

From $v(0) = 0$ we deduce $C_1 = 0$; From $v(\pi) = 0$ we get $C_2 \sin(\mu\pi) = 0$ which implies that $\mu = k$, $k \in \mathbb{N}$, for non-trivial solutions $v$. Then, the eigenvalues are $\lambda_k = -k^2$ and the eigenfunctions $v_k(x) = \sin(kx)$. Therefore for every $k \in \mathbb{N}$ and any $b_k \in \mathbb{R}$, the function $u_k(x,t) = b_k \sin(kx) e^{-ck^2 t}$ solves the problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad u(0,t) = u(\pi,t) = 0. \quad \text{(1 point.)}$$

Hence, by superposition, also $u(x,t) = \sum_{k=1}^{M} b_k \sin(kx) e^{-ck^2 t}$ solves the differential equation and the boundary conditions, for every $M \in \mathbb{N}$ and $b_1, ..., b_M \in \mathbb{R}$. The initial condition $u(x,0) = u_0(x)$ gives $b_k = a_k$ when $1 \leq k \leq N$ and $b_k = 0$ when $k \geq N+1$. Hence, $u = \sum_{k=1}^{N} a_k \sin(kx) e^{-ck^2 t}$ is the solution of the problem. (1 point.)

Note that for all $1 \leq k \leq N$, $\lim_{t\to\infty} u_k(x,t) = 0$ implies $\lim_{t\to\infty} u(x,t) = 0$ (and $\lim_{t\to\infty} \partial_t u_k(x,t) = 0$ implies $\lim_{t\to\infty} \partial_t u(x,t) = 0$), which shows that the temperature converges to 0 as $t \to \infty$. All the initial heat $u_0$ (which for physical reasons has to be assumed to be non-negative) flows out through the boundary which is kept at zero temperature. (1 point.)

**Problem 4:** (4+4* points.) See Theorem 3.4 (b) and Theorem 3.12 on the lecture notes.

**Problem 5:** (2 points.) (Can be solved by calculating $\Delta v$ or by showing directly the mean value property for $v$). For any $x \in \mathbb{R}^n$ and sufficiently small $r > 0$, since the spherical measure is rotation invariant

$$(Mv)(x,r) = \frac{1}{\omega_n}\int_S v(x+ry)\,d\sigma(y) = \frac{1}{\omega_n}\int_S u(T(x+ry))\,d\sigma(y)$$
$$\overset{Ty=z}{=} \frac{1}{\omega_n}\int_S u(Tx+rz)\,d\sigma(z) = u(Tx) = v(x). \quad \text{(2 points.)}$$

**Problem 6:** (3+2 points.) 1. By definition, for each $x \in \Omega$ and each sufficiently small $r > 0$ satisfying the side condition $\overline{B(x,r)} \subset \Omega$, $v$ satisfies

$$v(x) \le \frac{1}{\omega_n}\int_S v(x+ry)\,d\sigma(y). \quad \text{(1 point.)}$$

The above inequality holds true for any $0 < t < r$. We multiply the inequality by $t^{n-1}$ and integrate over $t$ from 0 to $r$ to derive

$$v(x)|B(x,r)| = v(x)\int_0^r\int_S 1\,d\sigma(y)t^{n-1}\,dt = \int_0^r v(x)\omega_n t^{n-1}\,dt \quad \text{(1 point.)}$$
$$\le \int_0^r\int_S v(x+ty)\,d\sigma(y)t^{n-1}\,dt = \int_{B(x,r)} v(y)\,dy. \quad \text{(1 point.)}$$

2. (Can be solved by calculating $\Delta(\phi(u))$, $\Delta|\nabla u|^2$ or by showing directly the mean value property using Jensen's inequality).

$$\frac{\partial}{\partial x_k}\phi(u) = \phi'(u)\frac{\partial u}{\partial x_k},\ \frac{\partial^2}{\partial x_k^2}\phi(u) = \phi''(u)\Big(\frac{\partial u}{\partial x_k}\Big)^2 + \phi'(u)\frac{\partial^2 u}{\partial x_k^2},$$

$$\Delta(\phi(u)) = \underbrace{\phi''(u)|\nabla u|^2}_{\ge 0} + \phi'(u)\underbrace{\Delta u}_{=0} \ge 0. \quad \text{(1 point.)}$$

$$|\nabla u|^2 = \sum_{i=1}^n \left(\frac{\partial u}{\partial x_i}\right)^2,\ \frac{\partial|\nabla u|^2}{\partial x_k} = 2\sum_{i=1}^n \frac{\partial u}{\partial x_i}\frac{\partial^2 u}{\partial x_i \partial x_k},\ \frac{\partial^2|\nabla u|^2}{\partial x_k^2} = 2\sum_{i=1}^n\left(\frac{\partial^2 u}{\partial x_i\partial x_k}\right)^2 + 2\sum_{i=1}^n \frac{\partial u}{\partial x_i}\frac{\partial^3 u}{\partial x_i\partial x_k^2},$$

$$\Delta|\nabla u|^2 = \underbrace{2\sum_{i,k=1}^n\left(\frac{\partial^2 u}{\partial x_i\partial x_k}\right)^2}_{\ge 0} + \underbrace{2\sum_{i=1}^n \frac{\partial u}{\partial x_i}\frac{\partial \Delta u}{\partial x_i}}_{=0} \ge 0. \quad \text{(1 point.)}$$

**Problem 7:** (2 points.) Notice that

71 (0.5) $$\frac{1}{(4\pi t)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-\frac{|x-\xi|^2}{4t}}\varphi(\xi)\,d\xi = \frac{1}{(4\pi t)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-\frac{|x-\xi|^2}{4t}}\psi(\xi)\,d\xi, \quad \text{(1 point.)}$$

and since $\psi$ is bounded, the RHS of (0.5) is a solution of the heat equation on $\mathbb{R}^n \times (0,\infty)$ with initial value $\psi$. Therefore, by using Theorem 4.1 of the lecture notes

$$\lim_{\substack{x\to x_0\\ t\to 0^+}} \frac{1}{(4\pi t)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-\frac{|x-\xi|^2}{4t}}\psi(\xi)\,d\xi = \psi(x_0). \quad \text{(1 point.)}$$

**Problem 8:** (4 points.) $a = x^2$, $b = 0$, $c = -y^2$ and $b^2 - ac = x^2y^2 > 0$, hence, the equation is hyperbolic on $(0,\infty)^2$. (1 point.)

The equation for the characteristics reads

$$x^2\dot{y}^2 - y^2\dot{x}^2 = 0, \quad \text{and} \quad \dot{y}^2 + \dot{x}^2 > 0 \quad (x > 0, y > 0).$$

We look for the solutions of the form $y = f(x)$ or $x = g(y)$. The above equation reduces to

$$x^2 f'(x)^2 - f(x)^2 = 0 \quad \text{or} \quad g(y)^2 - y^2 g'(y)^2 = 0.$$

There are two families of solutions for this problem:

$$y = c_1 x, \quad y = \frac{c_2}{x}. \quad \text{(1 point.)}$$

Introduce the new coordinates:

$$\xi = \frac{y}{x}, \quad \eta = xy.$$

We calculate

$$\frac{\partial u}{\partial x} = -\frac{y}{x^2}\frac{\partial u}{\partial \xi} + y\frac{\partial u}{\partial \eta}, \quad \frac{\partial^2 u}{\partial x^2} = \frac{y^2}{x^4}\frac{\partial^2 u}{\partial \xi^2} + \frac{2y}{x^3}\frac{\partial u}{\partial \xi} + y^2\frac{\partial^2 u}{\partial \eta^2} - 2\frac{y^2}{x^2}\frac{\partial^2 u}{\partial \xi \partial \eta},$$
$$\frac{\partial u}{\partial y} = \frac{1}{x}\frac{\partial u}{\partial \xi} + x\frac{\partial u}{\partial \eta}, \quad \frac{\partial^2 u}{\partial y^2} = \frac{1}{x^2}\frac{\partial^2 u}{\partial \xi^2} + 2\frac{\partial^2 u}{\partial \xi \partial \eta} + x^2\frac{\partial^2 u}{\partial \eta^2}.$$

We insert the result into the original equation to obtain

$$\frac{\partial^2 u}{\partial \xi \partial \eta}(\xi, \eta) = \frac{1}{2\eta}\frac{\partial u}{\partial \xi}(\xi, \eta), \quad \text{for} \quad (\xi, \eta) \in (0, \infty)^2. \quad \text{(2 points.)}$$

We set $v(\xi, \eta) = \frac{\partial u}{\partial \xi}(\xi, \eta)$, hence, $v(\xi, \eta)$ satisfies the equation

$$\frac{\partial v}{\partial \eta}(\xi, \eta) = \frac{1}{2\eta} v(\xi, \eta),$$

which has the general solution $v(\xi, \eta) = \sqrt{\eta} h_0(\xi)$, where $h_0$ is an arbitrary continuous function. Hence, $u(\xi, \eta)$ satisfies the equation $\frac{\partial u}{\partial \xi}(\xi, \eta) = \sqrt{\eta} h_0(\xi)$, which has the general solution

$$u(\xi, \eta) = \int^{\xi} \sqrt{\eta} h_0(s)\, ds + h_1(\eta) = \sqrt{\eta}\, \tilde{h}_0(\xi) + h_1(\eta),$$

where $h_1 \in C^1(\mathbb{R}^+)$ is arbitrary, and $\tilde{h}_0(\xi) := \int^{\xi} h_0(s)\, ds$.

Hence,

$$u(x, y) = \sqrt{xy}\, \tilde{h}_0\left(\frac{y}{x}\right) + h_1(xy).$$

(1 point.)