Karlsruhe Institue for Technology SS 2010
Institute for Analysis
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2. Exercise sheet - Computer assisted methods for partial differential equations

Exercise 4

Let $\Omega \in \mathbb{R}^n$, $n \leq 3$ be a bounded domain with Lipschitz boundary and $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ such that f and $\frac{\partial f}{\partial y}$ are continuous on $\overline{\Omega} \times \mathbb{R}$. Furthermore the operator $F : \mathcal{H} \to L^2(\Omega)$, where $\mathcal{H} = H^2(\Omega) \cap H_0^1(\Omega)$, is defined by

$$F[u](x) = -\Delta u(x) + f(x, u(x)) \quad (u \in \mathcal{H}).$$

Show that for $\tilde{u} \in \mathcal{H}$, F is Fréchet-differentiable in $\tilde{u}$ and that its Fréchet-derivative is given by $F'(\tilde{u}) = L$ with $L : \mathcal{H} \to L^2(\Omega)$,

$$L[u](x) = -\Delta u(x) + \frac{\partial f}{\partial y}(x, \tilde{u}(x))u(x) \quad (u \in \mathcal{H}).$$

Exercise 5

Let $\Omega \subset \mathbb{R}^2$ be a polygon and $\{\Omega_1, \ldots \Omega_N\}$ be a triangulation of Ω, i.e.

- Ω_i is an open triangle for all $i = 1, \ldots, N$
- $\overline{\Omega} = \bigcup_{i=1}^{N} \overline{\Omega}_i$
- $\Omega_i \cap \Omega_j = \emptyset \quad$ if $i \neq j$
- For $i \neq j$: $\overline{\Omega}_i \cap \overline{\Omega}_j$ is either $\emptyset$ *or* a complete edge of both triangles *or* a vertex of both triangles

Let $u : \overline{\Omega} \to \mathbb{R}$ be continuous and $u|_{\overline{\Omega}_i} \in C^2(\overline{\Omega}_i)$ $(i = 1, \ldots, N)$. Prove that the following equivalence is true:

$$u \in H^2(\Omega) \iff u \in C^1(\overline{\Omega})$$

Exercise 6

Let H be an infinite-dimensional Hilbert-space. Construct a continuous function $F : H \to \mathbb{R}$, which does not have the property that bounded sets are mapped into bounded sets.

Hint: Let $\{e_k\}_{k=1}^{\infty}$ be an orthonormal system in H. Define continuous functions $f_n : H \to \mathbb{R}$ such that $f_n(e_n) = n$ and $f_n(x) = 0$ for $x \in H$, $\|x - e_n\| \geq r$ ($r > 0$ appropriately chosen).

The exercise sheet will be discussed in the exercise on May, 21st.