Karlsruhe Institue for Technology SS 2010
Institute for Analysis
Prof. Dr. Michael Plum
Dipl.-Math. Dagmar Roth

3. Exercise sheet - Computer assisted methods for partial differential equations

Exercise 7

Let $\emptyset \neq \Omega \subset \mathbb{R}^4$ be an arbitrary domain. Prove, that there exists $\tilde{u} \in H^2(\Omega)$, such that $\tilde{u}$ has no continuous representative.

Exercise 8

Let $\Omega = (0,1) \subset \mathbb{R}$

a) Prove the following explicit embedding inequality for the embedding $H^1(\Omega) \hookrightarrow C(\bar{\Omega})$:

$$\|u\|_\infty \leq C_0 \|u\|_{L^2(\Omega)} + C_1 \|u'\|_{L^2(\Omega)}$$

where

$$C_0 \geq 1 \ \text{ arbitrary}, \quad C_1 = \frac{1}{\sqrt{3}C_0}.$$

Prove moreover, that the inequality is valid for

(i) $C_0 = 0,\ C_1 = \frac{1}{2}$ if $u(0) = u(1) = 0$

(ii) $C_0 = 0,\ C_1 = 1$ if $u(0) = 0$ or $u(1) = 0$

(iii) $C_0 \geq 1$ arbitrary, $C_1 = \frac{1}{2\sqrt{3}C_0}$ if $u(0) = u(1)$

b) Show, that any $u \in H^1(\Omega)$ has a continuous representative $u^* \in C(\bar{\Omega})$ and that the operator

$$E : \begin{cases} H^1(\Omega) & \to & C(\bar{\Omega}) \\ u & \mapsto & u^* \end{cases}$$

is compact.

Hint: Arzelà-Ascoli theorem

The exercise sheet will be discussed in the exercise on May, 28th.