5. Exercise sheet - Computer assisted methods for partial differential equations

In the following, let \((H, \langle \cdot, \cdot \rangle)\) be a Hilbert space and \(D(M) \subset H\) a linear subspace.

**Exercise 12**

Let \(M : D(M) \times D(M) \to H\) be a symmetric bilinear form such that the eigenelements \((u_i)_{i \in \mathbb{N}}\) of \(M(u, v) = \lambda \langle u, v \rangle\) (for all \(v \in D(M)\)) form an orthonormal basis of \(H\) and the eigenvalue sequence satisfies \(0 < \lambda_1 \leq \lambda_2 \leq \ldots\) and \(\lambda_i \to \infty\) \((i \to \infty)\). Prove the max-min-principle:

\[
\lambda_1 = \min_{u \in D(M) \setminus \{0\}} \frac{M(u, u)}{\langle u, u \rangle} \quad \lambda_{n+1} = \sup_{v_1, \ldots, v_n \in H} \inf_{u \in D(A) \setminus \{0\}} \frac{M(u, u)}{\langle u, u \rangle} \quad (n \in \mathbb{N})
\]

Here, \([v_1, \ldots, v_n]^\perp\) denotes the orthogonal complement of the subspace spanned by \(v_1, \ldots, v_n\).

**Exercise 13**

Consider the eigenvalue problem

\[
\begin{aligned}
(\ast) \quad \left\{ 
\begin{array}{ll}
-u''(x) & = \lambda (2 + \sin x) u(x) & \text{in } (0, \pi) \\
u(0) & = u(\pi) = 0 
\end{array}
\right.
\end{aligned}
\]

a) Give a weak formulation of problem \((\ast)\) using an appropriate bilinear form \(M\) defined on the space \(L^2((0, \pi); g)\) where the inner product is given by

\[
\langle u, v \rangle = \langle gu, v \rangle_{L^2}
\]

with \(g(x) = 2 + \sin x\).

b) Use the Rayleigh-Ritz procedure to compute an upper bound \(\tilde{\lambda}_1\) for the first eigenvalue of problem \((\ast)\).

*Hint:* Use \(\tilde{u}(x) = \sin x\) as trial function/approximate eigenfunction.

c) Compare \((\ast)\) with the eigenvalue problem

\[
\begin{aligned}
(\ast\ast) \quad \left\{ 
\begin{array}{ll}
-u''(x) & = 3 \mu u(x) & \text{in } (0, \pi) \\
u(0) & = u(\pi) = 0 
\end{array}
\right.
\end{aligned}
\]

to find a lower bound \(\rho\) for the second eigenvalue \(\lambda_2\) of \((\ast)\) which satisfies \(\tilde{\lambda}_1 < \rho\).

d) Use the Lehmann-procedure to compute a lower bound \(\underline{\lambda}_1\) for \(\lambda_1\).

Please turn over!
Exercise 14

In the following, let $A, A_1, A_2$ be real $n \times n$ matrices and $\Delta = I - A$.

a) Let $\|\Delta\|_\infty < 1$. Prove that $A$ is invertible and

$$\left\| A^{-1} - \sum_{k=0}^{m-1} \Delta^k \right\|_\infty \leq \frac{\|\Delta^m\|_\infty}{1 - \|\Delta\|_\infty}.$$ 

Here, $\| \cdot \|_\infty$ denotes the row-sum norm.

b) Let $A_1$ and $A_2$ be “almost diagonal” and consider the eigenvalue problem

$$(EV) \ A_1 x = \Lambda A_2 x \quad (\Lambda \in \mathbb{R}, \ x \in \mathbb{R}^n)$$

(consider for example matrices $(A_1)_{ij} = M(\tilde{u}_i, \tilde{u}_j), \ (A_2)_{ij} = \langle \tilde{u}_i, \tilde{u}_j \rangle$ where $\tilde{u}_1, \ldots, \tilde{u}_n$ are approximate eigenelements of the eigenvalue problem in exercise 12). Describe a procedure that gives eigenvalue enclosures for problem (EV) using the theorem of Gerschgorin.

The exercise sheet will be discussed in the exercise on June, 25th.