Karlsruhe Institute for Technology SS 2010
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5. Exercise sheet - Computer assisted methods for partial differential equations

In the following, let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space and $D(M) \subset H$ a linear subspace.

Exercise 12

Let $M : D(M) \times D(M) \to H$ be a symmetric bilinear form such that the eigenelements $(u_i)_{i\in\mathbb{N}}$ of $M(u,v) = \lambda\langle u,v\rangle$ (for all $v \in D(M)$) form an orthonormal basis of H and the eigenvalue sequence satisfies $0 < \lambda_1 \leq \lambda_2 \leq \ldots$ and $\lambda_i \to \infty$ $(i \to \infty)$. Prove the max-min-principle:

$$\begin{aligned} \lambda_1 &= \min_{\substack{u\in D(A)\\ u\neq 0}} \frac{M(u,u)}{\langle u,u\rangle} \\ \lambda_{n+1} &= \sup_{v_1,\ldots,v_n\in H} \inf_{\substack{u\in[v_1,\ldots,v_n]^\perp\\ u\in D(A),u\neq 0}} \frac{M(u,u)}{\langle u,u\rangle} \quad (n\in\mathbb{N}) \end{aligned}$$

Here, $[v_1,\ldots,v_n]^\perp$ denotes the orthogonal complement of the subspace spanned by $v_1,\ldots,v_n$.

Exercise 13

Consider the eigenvalue problem

$$(*) \begin{cases} -u''(x) &= \lambda(2+\sin x)u(x) \quad \text{in } (0,\pi) \\ u(0) &= u(\pi) = 0 \end{cases}$$

a) Give a weak formulation of problem $(*)$ using an appropriate bilinear form M defined on the space $L^2((0,\pi);g)$ where the inner product is given by

$$\langle u,v\rangle = \langle gu,v\rangle_{L^2}$$

with $g(x) = 2 + \sin x$.

b) Use the Rayleigh-Ritz procedure to compute an upper bound $\bar{\lambda}_1$ for the first eigenvalue of problem $(*)$.

Hint: Use $\tilde{u}(x) = \sin x$ as trial function/approximate eigenfunction.

c) Compare $(*)$ with the eigenvalue problem

$$(**) \begin{cases} -u''(x) &= 3\mu u(x) \quad \text{in } (0,\pi) \\ u(0) &= u(\pi) = 0 \end{cases}$$

to find a lower bound ρ for the second eigenvalue λ_2 of $(*)$ which satisfies $\bar{\lambda}_1 < \rho$.

d) Use the Lehmann-procedure to compute a lower bound $\underline{\lambda}_1$ for λ_1.

Please turn over!

Exercise 14

In the following, let A, A_1, A_2 be real $n \times n$ matrices and $\Delta = I - A$.

a) Let $\|\Delta\|_\infty < 1$. Prove that A is invertible and

$$\left\| A^{-1} - \sum_{k=0}^{m-1} \Delta^k \right\|_\infty \leq \frac{\|\Delta^m\|_\infty}{1 - \|\Delta\|_\infty}.$$

Here, $\|\cdot\|_\infty$ denotes the row-sum norm.

b) Let A_1 and A_2 be "almost diagonal" and consider the eigenvalue problem

$$\text{(EV)} \quad A_1 x = \Lambda A_2 x \quad (\Lambda \in \mathbb{R},\ x \in \mathbb{R}^n)$$

(consider for example matrices $(A_1)_{ij} = M(\tilde{u}_i, \tilde{u}_j)$, $(A_2)_{ij} = \langle \tilde{u}_i, \tilde{u}_j \rangle$ where $\tilde{u}_1, \dots, \tilde{u}_n$ are approximate eigenelements of the eigenvalue problem in exercise 12). Describe a procedure that gives eigenvalue enclosures for problem (EV) using the theorem of Gerschgorin.

The exercise sheet will be discussed in the exercise on June, 25th.