Karlsruhe Institute for Technology SS 2010
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6. Exercise sheet - Computer assisted methods for partial differential equations

Exercise 15

Let M be as in exercise 12 and moreover positive semidefinite. Let $u \in H$ be given and consider the problem to find some $w \in D(M)$ such that

$$(P) \quad M[w, v] = \langle u, v \rangle \quad (\text{for all } v \in D(M)).$$

a) Prove that $w \in D(M)$ is a solution of (P) if and only if

$$-M[w, w] = \min\{M[v, v] - 2\langle u, v \rangle : \ v \in D(M)\}$$

and the above minimum is taken for $v = w$.

b) Let X be a real vector space, $T : D(M) \to X$ a linear operator and $b : X \times X \to \mathbb{R}$ a symmetric, positive semidefinite bilinear form such that

$$M[\tilde{u}, \tilde{v}] = b[T\tilde{u}, T\tilde{v}] \quad (\tilde{u}, \tilde{v} \in D(M)).$$

Show that a solution of (P) satisfies

$$-M[w, w] = \max\{-b[g, g] : \ g \in V_u\},$$

where $V_u := \{g \in X : \ b[Tv, g] = \langle v, u \rangle \text{ for all } v \in D(M)\}$.

Please turn over!

Exercise 16

Let $\Omega \subset \mathbb{R}^2$ a bounded domain with C^1-boundary. Let $D(M) = \{u \in H^1(\Omega) : \int_\Omega u\,dx = 0\}$ and

$$M[u,v] = \int_\Omega \nabla u \cdot \nabla v\,dx \quad (u,v \in D(M)).$$

Consider the eigenvalue problem

$$M[u,v] = \lambda \langle u,v\rangle_{L^2} \quad (\text{for all } v \in D(M)).$$

What is the strong formulation of this eigenvalue problem?

Define now

$$\begin{aligned} X &:= D(M) \times L^2(\Omega) \\ Tu &:= (u,u) \in X \quad (u \in D(M)) \\ b\left[(\hat{u}_1,\hat{u}_2),(\hat{v}_1,\hat{v}_2)\right] &:= \int_\Omega \nabla \hat{u}_1 \cdot \nabla \hat{v}_1\,dx + \gamma \int_\Omega (\hat{u}_2\hat{v}_2 - \hat{u}_1\hat{v}_1)\,dx, \end{aligned}$$

where $\gamma \in \mathbb{R}$.

a) Prove, that b is positive semidefinite, if $\gamma > 0$ is chosen appropriately and that

$$M[u,v] = b[Tu,Tv] \quad \text{for all } u,v \in D(M).$$

b) Construct elements $\hat{w} \in X$ such that for given $u \in D(M)$

$$b[\hat{w},Tv] = \langle u,v\rangle \quad (\text{for all } v \in D(M)).$$

The exercise sheet will be discussed in the exercise on July, 9th.