6. Exercise sheet - Computer assisted methods for partial differential equations

Exercise 15

Let $M$ be as in exercise 12 and moreover positive semidefinite. Let $u \in H$ be given and consider the problem to find some $w \in D(M)$ such that

\[(P) \quad M[w, v] = \langle u, v \rangle \quad \text{for all } v \in D(M).\]

a) Prove that $w \in D(M)$ is a solution of $(P)$ if and only if

\[-M[w, w] = \min \{M[v, v] - 2\langle u, v \rangle : v \in D(M)\}\]

and the above minimum is taken for $v = w$.

b) Let $X$ be a real vector space, $T : D(M) \to X$ a linear operator and $b : X \times X \to \mathbb{R}$ a symmetric, positive semidefinite bilinear form such that

\[M[\tilde{u}, \tilde{v}] = b[T\tilde{u}, T\tilde{v}] \quad (\tilde{u}, \tilde{v} \in D(M)).\]

Show that a solution of $(P)$ satisfies

\[-M[w, w] = \max \{-b[g, g] : g \in V_u\},\]

where $V_u := \{g \in X : b[Tv, g] = \langle v, u \rangle \text{ for all } v \in D(M)\}$.

Please turn over!
Exercise 16

Let $\Omega \subset \mathbb{R}^2$ a bounded domain with $C^1$-boundary. Let $D(M) = \{ u \in H^1(\Omega) : \int_\Omega u\, dx = 0 \}$ and

$$M[u, v] = \int_\Omega \nabla u \cdot \nabla v\, dx \quad (u, v \in D(M)).$$

Consider the eigenvalue problem

$$M[u, v] = \lambda (u, v)_{L^2} \quad \text{(for all } v \in D(M)).$$

What is the strong formulation of this eigenvalue problem?

Define now

$$X := D(M) \times L^2(\Omega) \quad Tu := (u, u) \in X \quad (u \in D(M))$$

$$b[(\hat{u}_1, \hat{u}_2), (\hat{v}_1, \hat{v}_2)] := \int_\Omega \nabla \hat{u}_1 \cdot \nabla \hat{v}_1\, dx + \gamma \int_\Omega (\hat{u}_2 \hat{v}_2 - \hat{u}_1 \hat{v}_1)\, dx,$$

where $\gamma \in \mathbb{R}$.

a) Prove, that $b$ is positive semidefinite, if $\gamma > 0$ is chosen appropriately and that

$$M[u, v] = b[Tu, Tv] \quad \text{for all } u, v \in D(M).$$

b) Construct elements $\hat{w} \in X$ such that for given $u \in D(M)$

$$b[\hat{w}, Tv] = (u, v) \quad \text{(for all } v \in D(M)).$$

The exercise sheet will be discussed in the exercise on July, 9th.