Computer Assisted Proofs for Partial Differential Equations (SS 2012, KIT)

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Homotopy method (Plum 1990)

Base problem — Homotopy — Given problem

L_0 $\qquad$ L

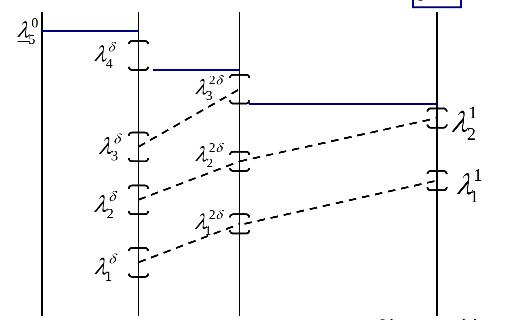
Explicit knowledge of eigenvalues

Homotopy

$L_s := (1-s)L_0 + sL$

$(s \in [0,1])$

■ Homotopy algorithm ■



[Nagatou, Nakao, Yamamoto (1999)]

Find $(u,\lambda) \in V \times \mathbb{C}$ s.t.

$$\begin{cases} Lu = \lambda u \\ \|u\| = 1 \end{cases}$$

Fixed Point Formulation → For $w=(u,\lambda)$

$$w = F(w)$$

(Fixed Point Equation)

$P: V \times \mathbb{C} \to V_N \times \mathbb{C}$ (projection)

$$\begin{cases} Pw = PF(w) & \cdots \text{ Newton's method} \\ (I-P)w = (I-P)F(w) & \cdots \text{ Norm estimation} \end{cases}$$

⇨ We construct a candidate set W suitable for Banach's fixed point theorem [Local uniqueness and simplicity]

Applications

Ex. 1: Allen-Cahn equation [Watanabe, Nakao (1993)]

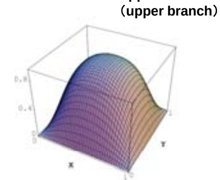
$$(1) \quad \begin{cases} -\Delta u = \lambda u(u-a)(1-u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

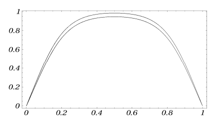
$\lambda = 150,\ a = 0.01,\ \Omega = (0,1)\times(0,1)$

ϕ_j : piecewise bilinear polynomials

The interval $(0,1)$ was partitioned into 80 pieces

Approx. sol. (upper branch)





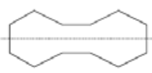
→ Algorithm 5.13

$\alpha^{(14)} = 0.0622$

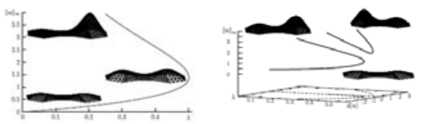
$y = 0.5$

Ex.2: Gelfand equation [Plum , Wieners (2002)]

(2) $\begin{cases} -\Delta u = \lambda e^u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$



$\lambda = 15/32$



$X = H_0^1(\Omega),\ Y = H^{-1}(\Omega)$

$\delta = 0.8979 \times 10^{-2}, \quad K = 3.126,$

$g(t) = \gamma t e^{t^2},\ \gamma := \left\| \lambda e^{\omega} \right\|_{L^4} \cdot 1.03 |\Omega|^{1/4} C_6^3$

$\left\| u^* - \omega \right\|_{H_0^1} \le 0.05066$

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Ex.3: Travelling waves in a nonlinearly supported beam

[Breuer, Horak, McKenna, Plum (2006)]

$u_{tt} + u_{xxxx} + e^u - 1 = 0, \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+$

⇓ Solitary waves $u = \varphi(x + ct)$

$\varphi^{iv} + c^2 \varphi'' + e^{\varphi} - 1 = 0$ (3)

> **Theorem:**
> For $c = 1.3$, Eq. (3) has at least 36 solutions.

$X = H_S^2(\mathbb{R}) := \{ u \in H^2(\mathbb{R}) : u(x) = u(-x) \text{ for all } x \in \mathbb{R} \},\ Y = H_S^{-2}$

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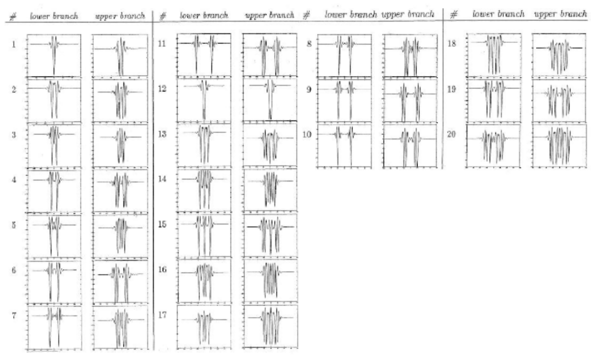

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Ex. 4: Spectrum of a 1-D Schroedinger operator

[Nagatou, Plum, Nakao (2011)]

We consider the following eigenvalue problem:

(4) $-u'' + q(x)u + s(x)u = \lambda u, \quad x \in \mathbb{R}$

- q : bounded, continuous, periodic
- $s \in L^\infty(\mathbb{R}),\ s(x) \to 0$ as $x \to \pm\infty$

⇓

Non-existence proof of point spectrum in a spectral gap

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Eigenvalue Excluding Method

Let Λ be a sufficiently narrow interval and consider a linear equation

$$(L - \Lambda)u = 0 \quad \text{on } \mathbb{R} \qquad (5)$$

Since it is clear that (5) has the trivial solution $u \equiv 0$, if we could validate the uniqueness of the solution of (5) by the method described below then it implies that any $\lambda \in \Lambda$ is NOT an eigenvalue of L, i.e. there is no eigenvalue in Λ.

> Computer assisted proof for the uniqueness of solution for (5)

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Uniqueness of the solution for $-u'' + q(x)u + s(x)u - \lambda u = 0$

⇑

Uniqueness of the solution for $u = F_\lambda(u)$

$F_\lambda(u) = \int_{\mathbb{R}} G(x,y,\lambda) s(y) u(y) dy$

$G(x,y,\lambda) = \begin{cases} \psi_1(x)\psi_2(y) & (x \le y) \\ \psi_2(x)\psi_1(y) & (x \ge y) \end{cases}$

⇑

Fundamental solutions ψ_1, ψ_2 of $-\psi'' + q(x)\psi - \lambda\psi = 0$

⇑

ϕ_1, ϕ_2 in $[0, r]$ $\quad \begin{pmatrix} \phi_1(0) = 1, & \phi_1'(0) = 0 \\ \phi_2(0) = 0, & \phi_2'(0) = 1 \end{pmatrix}$

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$q(x) = a \cdot \cos(2\pi x),\ s(x) = c \cdot e^{-x^2}$

The interval Λ was subdivided into narrow subintervals Λ_k whose widths are $0.001 \sim 0.01$.

Eigenvalue-free intervals

	Λ	The first spectral gap
$a=3, c=1$	$[8.822, 10.585]$	$[8.341644, 11.340563]$
$a=3, c=2$	$[9.0, 9.35]$	$[8.341644, 11.340563]$
$a=5, c=1$	$[7.55, 11.335]$	$[7.292924, 12.287917]$