Notes on

Lecture (01053400) - Dispersive Equations

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These are short incomplete notes, only for participants of the course Lecture (01053400) at the Karlsruhe Institute for Technology, Winter Term 2018/2019. Corrections are welcome to be sent to

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The following textbooks/lecture notes/articles are recommended:

- T. Cazenave: Semilinear Schrödinger equations.
- F. Linares, G. Ponce: Introduction to nonlinear dispersive equations.
- T. Tao: Nonlinear dispersive equations - local and global analysis.
- J. Colliander, M. Keel, G. Staffilani, H. Takaoka, T.Tao: The theory of nonlinear Schrödinger equations.
- H. Koch, D. Tataru: Conserved energies for the cubic NLS in 1-d.

If you have more questions about the lecture, you are welcome to come directly to

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1 Introduction

In this lecture we will mainly consider the Cauchy problem for the semilinear Schrödinger equation

$$\begin{cases} i\partial_t u + \Delta u = \kappa |u|^{p-1} u, \\ u|_{t=0} = u_0(x). \end{cases} \tag{NLS}$$

Here $u = u(t,x) \in \mathbb{C}$ denotes the unknown wave function and $t \in \mathbb{R}$, $x \in \mathbb{R}^d$, $d \geqslant 1$ denote the time and space variables respectively. κ takes value in $\{\pm 1\}$ and we call the nonlinear Schrödinger equation in (NLS) defocusing if $\kappa = 1$ (repulsive nonlinearity) and focusing if $\kappa = -1$ (attractive nonlinearity) respectively. $p > 1$ is a real constant which plays an important role in the mathematical theory and if $p = 3$ we call (NLS) the cubic nonlinear Schrödinger equation.

We will consider the wellposedness issue of this Cauchy problem (NLS), the asymptotic behaviour of the solutions (scattering, blowup, solitons, etc.) and the conserved energies for the completely integrable case. We will see that the results will depend heavily on the space dimension d, the sign of κ and the nonlinearity exponent p. We will also pay much attention to the functional space where the initial data u_0 stays in.

The (NLS) equation is a semilinear dispersive equation, which possesses symmetry structures, conservation laws, and special solutions (e.g. solitary wave solutions when $\kappa = -1$). We are going to explain these basic concepts in Subsection 1.1.

1.1 Basic concepts

We are going to explain (heuristically) some basic concepts related to (NLS), such as dispersion, semilinearity, symmetries, conservation laws, solitary waves, in this subsection.

1.1.1 Dispersion

What does dispersion mean? Is the equation (NLS) dispersive? Roughly speaking, the dispersion means that "Waves with different frequencies travel at different velocities" and the dispersion property is related to the linear part of (NLS).

We now give some formal explanation. Let $d = 1$ and let $u(x,t)$ be a plane-wave solution

$$u(t,x) = e^{i(kx-\omega t)} = e^{ik(x-(\omega/k)t)}$$

of the linear Schrödinger equation $iu_t + u_{xx} = 0$, where k is the wave number (waves per unit length) and ω denotes the (angular) frequency. Then we derive the dispersion relation $\omega = \omega(k) = k^2$, such that $u(t,x)$ is a travelling wave with the phase velocity $c(k) = \omega(k)/k = k$ and the larger k is, the faster the wave travels, that is, high frequency waves travel much faster than low frequency waves! More generally, we take the inverse Fourier transform of the initial data

$$u_0(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ikx} \widehat{u_0}(k) dk,$$

then by superposition the solution of the linear Schrödinger equation reads (noticing that $e^{ik(x-c(k)t)}$ is the solution with initial data e^{ikx})

$$u(t,x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ik(x-c(k)t)} \widehat{u_0}(k) dk.$$

The fact that various Fourier modes travel at different speeds is considered to be dispersive phenomenon mathematically.

The well known Korteweg-de Vries (KdV) equation is also a nonlinear dispersive equation:

$$\partial_t u + u_{xxx} + uu_x = 0, \quad u|_{t=0} = u_0. \tag{KdV}$$

The linear part $u_t + u_{xxx} = 0$ has the phase velocity $c(k) = -k^2$.

There is an obvious example which is not a dispersive equation:

$$\partial_t u + c\partial_x u = 0, \quad u|_{t=0} = u_0, \text{ where } c = \textit{constant}.$$

The solution $u(t,x) = u_0(x - ct)$ travels at constant speed c.

1.1.2 Semilinearity

The nonlinear Schrödinger equation (NLS) is of the semilinear form:

$$i\partial_t u + \Delta u = f(u), \text{ i.e. } \partial_t u = i\Delta u - if(u), \quad u|_{t=0} = u_0, \tag{1.1}$$

where the function f depends nonlinearly only on lower order terms: u (not on $\partial_t u, \nabla^2 u$)!

Recall the ODE theory. Consider the ODE of the form

$$v' = Lv + f(v), \quad v|_{t=0} = v_0,$$

where $v : \mathbb{R} \mapsto \mathbb{R}^d$ is the unknown, $t \in \mathbb{R}$ denotes the time variable, $L \in M_d(\mathbb{R})$ some linear transform and $f : \mathbb{R}^d \mapsto \mathbb{R}^d$ some function. We have the Duhamel formula for the solution

$$v(t) = e^{tL}v_0 + \int_0^t e^{(t-t')L} f(v(t'))\, \mathrm{d}t' .$$

Now we take the Fourier transform in x-variable to the equation (1.1)

$$\partial_t \widehat{u}(t,\xi) = -i|\xi|^2 \widehat{u}(t,\xi) - i\widehat{f(u)}(\xi), \quad \widehat{u}(0,\xi) = \widehat{u_0}(\xi).$$

View ξ as a parameter, then we also have (at least formally) the Duhamel formula for $\widehat{u}(\cdot;\xi)$

$$\widehat{u}(t,\xi) = e^{-it|\xi|^2}\widehat{u_0}(\xi) - i\int_0^t e^{-i(t-t')|\xi|^2}\widehat{f(u)}(t',\xi)\, \mathrm{d}t' .$$

We calculate the inverse Fourier transform of the tempered distribution $e^{-it|\xi|^2}$ (t viewed as parameter) now. We consider $e^{-it|\xi|^2}$ as the limit of the Schwartz functions $e^{-\varepsilon|\xi|^2 - it|\xi|^2}$ as $\varepsilon \to 0$ pointwisely (and hence in the tempered distribution sense). Then recalling

$$\mathcal{F}^{-1}(e^{-\frac{1}{2}|\xi|^2}) = e^{-\frac{1}{2}|x|^2}, \quad \mathcal{F}^{-1}(g(a\xi)) = a^{-d}\mathcal{F}^{-1}(g)(a^{-1}x),$$

$\mathcal{F}^{-1}(e^{-it|\xi|^2})$ is the limit of $(2(\varepsilon+it))^{-\frac{d}{2}}e^{-\frac{|x|^2}{4(\varepsilon+it)}}$: $(2it)^{-\frac{d}{2}}e^{-\frac{|x|^2}{4it}}$. Recalling also

$$\mathcal{F}^{-1}(\widehat{g}\widehat{h}) = (2\pi)^{-\frac{d}{2}} g * h,$$

we derive from the above Duhamel formula that

$$u(t,x) = K_t * u_0 - i\int_0^t K_{t-s} * f(u(s,\cdot))ds, \quad K_t := (4\pi i t)^{-\frac{d}{2}} e^{i\frac{|\cdot|^2}{4t}}.$$

In other words (or by Stone's theorem), for the selfadjoint operator Δ on the Hilbert space $H = L^2(\mathbb{R}^d)$ with the domain $D(\Delta) = H^2(\mathbb{R}^d)$, there exists a unique strongly continuous unitary group $S(t) = e^{it\Delta} := K_t *$ on H such that

$$\frac{d}{dt}\Big|_{t=0} S(t)\phi = i\Delta\phi, \quad \forall \phi \in D(\Delta) = H^2(\mathbb{R}^d).$$

We rewrite the above Duhamel formula as

$$u(t,x) = S(t)u_0 - i\int_0^t S(t-s)f(u(s,\cdot))ds, \qquad \text{(Duhamel)}$$

where we denote $S(t)$ as the following unitary group on $L^2(\mathbb{R}^d)$:

$$S(t)g = e^{it\Delta}g = K_t * g = (4\pi it)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{i\frac{|x-y|^2}{4t}} g(y)dy. \tag{St}$$

It is obvious from (St) that

$$\|S(t)u_0\|_{L^\infty(\mathbb{R}^d)} \leqslant |4\pi t|^{-\frac{d}{2}} \|u_0\|_{L^1(\mathbb{R}^d)}, \tag{1.2}$$

which implies that if $u_0 \in L^1(\mathbb{R}^d)$, then the solution $S(t)u_0$ of the linear Schrödinger equation decays of rate $|t|^{-\frac{d}{2}}$ as $|t| \to \infty$. This is exactly the dispersion phenomenon. Since $S(t)$ is unitary, then

$$\|S(t)u_0\|_{L^2(\mathbb{R}^d)} = \|u_0\|_{L^2(\mathbb{R}^d)}. \tag{1.3}$$

Recall the Riesz-Thorine interpolation theorem:

Theorem 1.1. *Let $1 \leqslant p_0 \neq p_1 \leqslant \infty$, $1 \leqslant q_0 \neq q_1 \leqslant \infty$. If $T \in \mathcal{L}(L^{p_j}(\mathbb{R}^d), L^{q_j}(\mathbb{R}^d))$ be the linear operator from $L^{p_j}(\mathbb{R}^d)$ to $L^{q_j}(\mathbb{R}^d)$ with the operator norm M_j, $j = 0, 1$, then for any $0 \leqslant \theta \leqslant 1$,*

$$T \in \mathcal{L}(L^{p_\theta}(\mathbb{R}^d), L^{q_\theta}(\mathbb{R}^d)), \quad \frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \ \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1},$$

with the operator norm $M_\theta \leqslant M_0^{1-\theta} M_1^\theta$.

Hence we derive from (1.2)-(1.3) that

Proposition 1.1. *Let $S(t)$ be the unitary map defined by (St). Then $S(t)$ is a linear map from $L^{r'}(\mathbb{R}^d)$ to $L^r(\mathbb{R}^d)$ for any $r \in [2, \infty]$ (with $\frac{1}{r} + \frac{1}{r'} = 1$) such that*

$$\|S(t)u_0\|_{L^r(\mathbb{R}^d)} \leqslant |4\pi t|^{-\frac{d}{2}+\frac{d}{r}} \|u_0\|_{L^{r'}(\mathbb{R}^d)}, \quad \forall t \neq 0. \tag{1.4}$$

1.1.3 Symmetries

We list here some interesting symmetries for the Schrödinger equation (NLS):

- Time/Space translation symmetry: If $u(t, x)$ solves (NLS), then $u_{t_0,x_0}(t, x) = u(t - t_0, x - x_0)$, $t_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}^d$ also solves the Schrödinger equation in (NLS);

- Space rotation symmetry: If $u(t, x)$ solves (NLS), then $u(t, \Omega x)$, $\Omega \in SO(d)$ also solves the Schrödinger equation in (NLS);

- Phase rotation symmetry: If $u(t, x)$ solves (NLS), then $e^{i\omega}u(t, x)$, $\omega \in \mathbb{R}$ also solves the Schrödinger equation in (NLS);

- Time reversal symmetry: If $u(t,x)$ solves (NLS), then $\bar{u}(-t,x)$ ($\bar{u}$ means the complex conjugate of u) also solves the Schrödinger equation in (NLS);
- Galilean invariance: If $u(t,x)$ solves (NLS), then $e^{i(x\cdot v-|v|^2 t)}u(t,x-2vt)$, $v\in\mathbb{R}^d$ also solves the Schrödinger equation in (NLS);
- Pseudo-conformal symmetry for the mass critical case $p=1+\frac{4}{d}$: If $u(t,x)$ solves (NLS), then $\frac{e^{i\frac{|x|^2}{4t}}}{t^{\frac{d}{2}}}u(-\frac{1}{t},\frac{x}{t})$, $\frac{e^{i\frac{|x|^2}{4(1+t)}}}{(1+t)^{\frac{d}{2}}}u(\frac{t}{1+t},\frac{x}{1+t})$, etc. $t>0$ also solve the Schrödinger equation in (NLS);
- Scaling symmetry: If $u(t,x)$ solves (NLS), then $u_\lambda(t,x)=\frac{1}{\lambda^{\frac{2}{p-1}}}u(\frac{t}{\lambda^2},\frac{x}{\lambda})$, $0\neq\lambda\in\mathbb{R}$ also solves the Schrödinger equation in (NLS).

[17.10.2018]

[19.10.2017]

Let us focus on the scaling symmetry for a while: Notice also that the scaling includes both linear and nonlinear informations in the nonlinear Schrödinger equation (NLS). Recall that the Sobolev space $H^s(\mathbb{R}^d)$, $s\in\mathbb{R}$ is defined to be

$$H^s(\mathbb{R}^d):=\{f\in\mathcal{S}'(\mathbb{R}^d)\mid \|f\|^2_{H^s(\mathbb{R}^d)}:=\int_{\mathbb{R}^d}(1+|\xi|^2)^s|\hat{f}(\xi)|^2\,\mathrm{d}\xi<\infty\},\tag{1.5}$$

and we also define the homogeneous Sobolev norm as

$$\|f\|^2_{\dot{H}^s(\mathbb{R}^d)}:=\int_{\mathbb{R}^d}|\xi|^{2s}|\widehat{f}(\xi)|^2 d\xi.\tag{1.6}$$

Denote the critical exponent

$$s_c=\frac{d}{2}-\frac{2}{p-1}.\tag{1.7}$$

Let the initial data $u_0\in H^s(\mathbb{R}^d)$, then the rescaled initial datum $u_{0,\lambda}(x)=\frac{1}{\lambda^{\frac{2}{p-1}}}u_0(\frac{x}{\lambda})$, $\lambda>0$ has $\dot{H}^s(\mathbb{R}^d)$-norm as follows

$$\|u_{0,\lambda}\|_{\dot{H}^s(\mathbb{R}^d)}=\lambda^{-s+s_c}\|u_0\|_{\dot{H}^s(\mathbb{R}^d)}.$$

Heuristically, we then divide the regularity exponent s of the Sobolev space H^s into three cases:

- $s > s_c$ (subcritical case)
  As $\lambda \to \infty$, $\|u_{0,\lambda}\|_{\dot{H}^s(\mathbb{R}^d)} \to 0$ and if the solution u exists on the time interval $[0, T_*]$ then the rescaled solution u_λ exists on the time interval $[0, \lambda^2 T_*]$ with $\lambda^2 T_* \to \infty$. This is the most favourable situation in well-posedness issue: we can make both the small initial norm and the long time interval at the same time.

- $s = s_c$ (critical case)
  It is easy to see that the $\dot{H}^{s_c}$-norm is invariant under the scaling: $\|u_{0,\lambda}\|_{\dot{H}^{s_c}(\mathbb{R}^d)} = \|u_0\|_{\dot{H}^{s_c}(\mathbb{R}^d)}$, and as $\lambda \to \infty$ the rescaled existing time interval is still $[0, \lambda^2 T_*]$ with $\lambda^2 T_* \to \infty$. This is always a unclear situation.

- $s < s_c$ (supercritical case)
  In this case as $\lambda \to \infty$, $\|u_{0,\lambda}\|_{\dot{H}^s(\mathbb{R}^d)} \to \infty$ as $\lambda^2 T_* \to \infty$, that is, the growing norm corresponds to longer time interval. Blowup may happen in this situation.

In particular, we are in the L^2 (mass)-subcritical/critical/supercritical case if

$$0 = s > / = / < s_c, \text{ i.e. } p < / = / > 1 + \frac{4}{d},$$

and we are in the H^1 (energy)-subcritical/critical/supercritical case if

$$1 = s > / = / < s_c, \text{ i.e. } p < / = / > 1 + \frac{4}{d-2}.$$

In this lecture we take the convention that $1 + \frac{4}{d-2} = \infty$ if $d = 1, 2$. It seems that we should have well-posedness results in L^2 or H^1 framework when $p < 1 + \frac{4}{d}$ or $p < 1 + \frac{4}{d-2}$ and we will indeed prove this in Section 2.

1.1.4 Conservation laws

Suppose that $u(t, x)$ is a smooth and fast decaying solution of the Schrödinger equation in (NLS). We have the following conservation laws a priori:

- Mass conservation law

$$M(u)(t) = \int_{\mathbb{R}^d} |u(t, x)|^2 \, \mathrm{d}x = M(u)(0). \tag{1.8}$$

  Indeed, we test this Schrödinger equation by $\bar{u}$ and then take the imaginary part to obtain (1.8). (**Exercise**)

- Momentum conservation law

$$P_j(u)(t) = \operatorname{Im} \int_{\mathbb{R}^d} \bar{u}\partial_{x_j} u \, \mathrm{d}x \; = P_j(u)(0). \tag{1.9}$$

  Indeed, we test the Schrödinger equation by $\partial_{x_j}\bar{u}$ and then take the real part to arrive at (1.9). (**Exercise**)

- Energy conservation law

$$E(u)(t) = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla u|^2 \, \mathrm{d}x \; + \frac{\kappa}{p+1}\int_{\mathbb{R}^d} |u|^{p+1} \, \mathrm{d}x \; = E(u)(0). \tag{1.10}$$

  Indeed, we test the Schrödinger equation by $\Delta\bar{u} - \kappa|u|^{p-1}\bar{u}$ and then take the imaginary part to get (1.10). (**Exercise**)

Remark 1.1. *The equation* (NLS) *is the nonlinear Hamiltonian flow of the Hamiltonian $E(u)$ with respect to the symplectic form ω:*

$$\omega(f, g) = \operatorname{Im} \int_{\mathbb{R}^d} f(x)\overline{g(x)} \, \mathrm{d}x \, .$$

That is to say, we can write (NLS) *as $\partial_t u = \nabla_\omega E(u)$, with $\nabla_\omega E$ defined by $\omega(f, \nabla_\omega E) = dE(f)$.*

Here the phase space $\mathcal{M}$ is an infinite-dimensional real linear space with the canonically conjugate coordinates (indexed by $x \in \mathbb{R}^d$) $q(x) : u \mapsto \operatorname{Re} u(x) \in \mathcal{S}(\mathbb{R}^d)$, $p(x) : u \mapsto \operatorname{Im} u(x) \in \mathcal{S}(\mathbb{R}^d)$. Define the Poisson bracket associated to ω as follows:

$$\{G, F\}(u) = \int_{\mathbb{R}^d} \frac{\delta F}{\delta p}\Big|_u(x)\frac{\delta G}{\delta q}\Big|_u(x) - \frac{\delta F}{\delta q}\Big|_u(x)\frac{\delta G}{\delta p}\Big|_u(x) \, \mathrm{d}x \, .$$

Then we can check that $\{M, E\} = \{P_j, E\} = \{E, E\} = 0$. Hence $M(u)$, $P_j(u)$ are conservation laws for the Hamiltonian flow (NLS).

Remark 1.2. *These above conservation laws obviously hold true for smooth and fast decaying solutions for* (NLS). *Nevertheless they can also hold for less regular solutions by the approximation argument, e.g. the mass conservation law* (1.8) *will hold for L^2-subcritical case with L^2 initial data. These conservation laws will help us to get global-in-time well-posedness result, see Subsection 2.2.2 below.*

1.1.5 Solitary wave

"A solitary wave is a wave that travels at a constant velocity without changing its shape." Specially, let $e^{it}Q(x)$ be a solitary wave of the Schrödinger equation (NLS), with $Q(x)$ satisfying the elliptic equation

$$\Delta Q - Q = \kappa |Q|^{p-1}Q, \quad \kappa = -1, \quad Q \in H^1(\mathbb{R}^d). \tag{1.11}$$

Then by symmetries in Subsection 1.1.3 we know that the following general solitary wave solution travels along the line $x = x_0 + 2vt$:

$$e^{i\lambda^{-2}t + ix\cdot v - i|v|^2 t + i\theta} Q_\lambda(x - x_0 - 2vt), \quad \theta \in \mathbb{R},\ x_0 \in \mathbb{R}^d,\ v \in \mathbb{R}^d, \tag{1.12}$$

with $Q_\lambda(x) = \frac{1}{\lambda^{\frac{2}{p-1}}} Q(\frac{x}{\lambda})$, $0 \neq \lambda \in \mathbb{R}$. We have taken $\kappa = -1$ the focusing case, otherwise in the defocusing case there exists only trivial solution: We test (1.11) with $\kappa = 1$ by $\bar{Q}$ to get

$$0 \geqslant -\int_{\mathbb{R}^d} |\nabla Q|^2 \,\mathrm{d}x = \int_{\mathbb{R}^d} (1 + |Q|^2)|Q|^2 \,\mathrm{d}x \geqslant 0 \Rightarrow Q = 0 \in H^1(\mathbb{R}^d).$$

Let $d = 1$, then (1.11) is an ODE and one has an explicit solution (unique up to translation and sign change)

$$Q(x) = \Big(\frac{p+1}{2} \operatorname{sech}^2\big(\frac{p-1}{2}x\big)\Big)^{\frac{1}{p-1}} \in H^1(\mathbb{R}^1), \quad \operatorname{sech}(x) = \frac{2}{e^x + e^{-x}}. \tag{1.13}$$

We will see that for any $d \geqslant 2$, in the energy-subcritical case (i.e. $1 < p < 1 + \frac{4}{d-2}$), there exists a unique *positive radial* H^1 solution (up to translation) of (1.11) in Proposition 4.5, Subsection 4.1. This unique solution is called the *ground state* and the corresponding solution $u(t,x) = e^{it}Q$ of (NLS) is the ground state standing wave and is often called *soliton*. We will show the orbital stability result of the solitons in the mass-subcritical case (i.e. $1 < p < 1 + \frac{4}{d}$) in Subsection 4.3. There are also other solutions (not necessarily positive or radial) than the ground state for (1.11) when $d \geqslant 2$ which are called bound states and we will not discuss them in this lecture.

Remark 1.3. *It is easy to see that for any $r \in [1, \infty]$, the L^r-norm of the initial datum Q is preserved by the solution $e^{it}Q$:*

$$\|e^{it}Q\|_{L^r(\mathbb{R}^d)} = \|Q\|_{L^r(\mathbb{R}^d)}, \ \forall t \in \mathbb{R}\,.$$

This phenomenon is totally different from the linear Schrödinger equation where the estimate (1.4) shows that $\|S(t)u_0\|_{L^r(\mathbb{R}^d)}$, $r > 2$ vanishes as $t \to \infty$. Hence the existence of the solitary waves describes a balance between the (linear) dispersion and the nonlinearity and they neither decay nor develop singularities.

[19.10.2017]

[26.10.2017]

1.2 Completely integrable case

In this section we take $d = 1$ and $p = 3$ in the equation of (NLS): the cubic nonlinear Schrödinger equation (by scaling $u \mapsto \frac{1}{\sqrt{2}}u$)

$$i\partial_t u + u_{xx} = 2\kappa|u|^2 u, \quad u|_{t=0} = u_0. \tag{1.14}$$

We are going to consider the defocusing case $\kappa = 1$ and study this cubic NLS via its Lax-pair formulation. We view u as the potential in the Lax operator (see (1.22) below) and hope to solve u by use of the scattering data defined on the real line associated to this Lax operator. We always assume sufficiently decay condition on u as $|x| \to \infty$, say $u \in \mathcal{S}(\mathbb{R})$.

Let $\kappa = 1$. By [Zakharov-Shabat 1972], the cubic nonlinear Schrödinger equation (1.14) can be viewed as the compatibility condition of the two systems

$$\psi_x = \begin{pmatrix} -iz & u \\ \bar{u} & iz \end{pmatrix} \psi := U\psi, \tag{1.15}$$

$$\psi_t = i\begin{pmatrix} -2z^2 - |u|^2 & -2izu + u_x \\ -2iz\bar{u} - \bar{u}_x & 2z^2 + |u|^2 \end{pmatrix} \psi := V\psi, \tag{1.16}$$

where in these two systems z is a parameter, (t, x) are space and time variables, u is some known function and $\psi : \mathbb{R} \times \mathbb{R} \to \mathbb{C}^2$ is the unknown vector-valued function. Here the compatibility condition of the above two systems means that

$$\begin{aligned} \psi_{xt} &= (U\psi)_t = U_t\psi + U\psi_t = U_t\psi + UV\psi \\ \text{and } \psi_{tx} &= (V\psi)_x = V_x\psi + V\psi_x = V_x\psi + VU\psi \end{aligned}$$

should be the same, that is,

$$U_t = V_x + [V, U] \text{ with } [V, U] := VU - UV. \tag{1.17}$$

We can check that (1.14) is exactly the compatibility condition (1.17).

Equations (1.15), (1.16) and their compatibility condition (1.17) have a natural geometric interpretation. Indeed, the matrix function $U(x, t, z)$ and $V(x, t, z)$ may be considered as local connection coefficients in the trivial vector bundle $\mathbb{R}^2 \times \mathbb{C}^2$ where the space-time $\mathbb{R}^2$ is the base and the vector-valued function $\psi(x, t, z)$ takes values in the fiber $\mathbb{C}^2$ (z can be viewed as a parameter). Equations (1.15), (1.16) show that ψ is a covariantly constant vector while (1.17) says that the (U, V)-connection has zero curvature. Hence a representation of a nonlinear equation in the form (1.17) is called a *zero curvature representation.*

We correspondingly have the Lax-pair formulation of cubic NLS (1.14). We rewrite the system (1.15) into the form of the spectral problem of the self-adjoint Lax operator L (with the domain depending on the potential u, e.g. $D(L) = H^1(\mathbb{R}) \subset L^2(\mathbb{R})$ if $u \in L^\infty(\mathbb{R})$) as follows

$$L\psi = z\psi, \quad L = \begin{pmatrix} i\partial_x & -iu \\ i\bar{u} & -i\partial_x \end{pmatrix}. \tag{1.18}$$

Then we can replace $z\psi$ by $L\psi$ in the system (1.16) to get

$$\begin{aligned} \psi_t = P\psi, \quad P &= 2i \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} L^2 + 2 \begin{pmatrix} 0 & u \\ \bar{u} & 0 \end{pmatrix} L + i \begin{pmatrix} -|u|^2 & u_x \\ -\bar{u}_x & |u|^2 \end{pmatrix} \\ &= i \begin{pmatrix} 2\partial_x^2 - |u|^2 & -u\partial_x - \partial_x u \\ \bar{u}\partial_x + \partial_x \bar{u} & -2\partial_x^2 + |u|^2 \end{pmatrix}. \end{aligned} \tag{1.19}$$

The compatibility condition (1.17), i.e. the cubic nonlinear Schrödinger equation (1.14) equals to the following evolutionary equation

$$L_t = [P, L], \tag{1.20}$$

and the operator pair (L, P) is called *Lax pair* for (1.14).

Formally, thanks to the evolutionary equation (1.20), if $L\psi = z\psi$ then z is independent of the time since we derive $z_t = 0$ from $(L\psi)_t = (z\psi)_t$:

$$\begin{aligned} (L\psi)_t &= L_t\psi + L\psi_t = [P, L]\psi + LP\psi = PL\psi, \\ (z\psi)_t &= z_t\psi + z\psi_t = z_t\psi + zP\psi = z_t\psi + PL\psi. \end{aligned}$$

This fact is non trivial since L depends on $u(t, x)$ and hence its spectrum should depend on t generally. More precisely we will indeed consider $z \in \mathbb{R}$ (the continuous spectrum of the self-adjoint operator L) and define the *transmission coefficient* $T(z)$ and the *reflection coefficient* $R(z)$ associated to the Lax operator L. We will see that $T(z)$ does not depend on time and hence gives infinite many conservation laws for the cubic NLS (1.14). On the other side, the Lax pair (1.20) gives a simple evolutionary equation for the scattering data $R(t, z)$ associated to $u(t, x)$: $\partial_t R(t, z) = 4iz^2 R(t, z)$. We will use the direct transform to get the initial scattering data $R_0(z)$ from the initial data $u_0(x)$ and then use the inverse scattering transform to get the solution $u(t, x)$ from the evolved scattering data $R(t, z)$. The direct/inverse scattering transforms give an algorithmic way to solve (NLS). This idea can be compared with the resolution of the linear Schrödinger equation via Fourier and inverse Fourier transform:

$$\begin{aligned}
&i\partial_t u + u_{xx} = 0, \quad u|_{t=0} = u_0, \\
&\Rightarrow i\partial_t \hat{u}(\xi) - \xi^2 \hat{u}(\xi) = 0, \quad \hat{u}|_{t=0} = \widehat{u_0}(\xi), \\
&\Rightarrow \hat{u}(t,\xi) = e^{-i\xi^2 t}\widehat{u_0}(\xi) \Rightarrow u(t,x) = \mathcal{F}_x^{-1}(\hat{u}).
\end{aligned} \tag{1.21}$$

The direct and inverse scattering transform play the same role of Fourier and inverse Fourier transform here, with the scattering data $R(t,z)$ viewed as the *nonlinear* Fourier transform of $u(t,x)$. However the direct and inverse scattering transform are nonlinear and much involved since it is related to the *nonlinear* Schrödinger equation.

1.2.1 Direct scattering transform

Let $z \in \mathbb{R}$. In the direct transform step, given the function $u = u(x) \in L^1(\mathbb{R};\mathbb{C})$, we consider the ODE system (1.15):

$$\psi_x = \begin{pmatrix} -iz & u \\ \bar{u} & iz \end{pmatrix} \psi, \tag{1.22}$$

where $x \in \mathbb{R}$ is the space variable and $\psi : \mathbb{R} \mapsto \mathbb{C}^2$ is the unknown vector-valued function.

We propose *reasonable* boundary condition (initial value condition) for the system (1.22). As the known function $u(x)$ decays sufficiently as $|x| \to \infty$, $\psi(x)$ can be approximated by the solution of (1.22) when $u \equiv 0$ at infinity:

$$\psi(x) = c_1 \begin{pmatrix} e^{-izx} \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ e^{izx} \end{pmatrix} + o(1), \quad |x| \to \infty, \tag{1.23}$$

where c_1, c_2 are arbitrary constants. Since $z \in \mathbb{R}$, these solutions oscillate at infinity. This is what defines $\mathbb{R}$ as the continuous spectrum for (1.22).

<u>Jost solutions</u>

We will get in the Appendix A the unique solutions $j^{-,1}, j^{-,2}$ for the initial value problem (1.22) with the following initial value conditions respectively:

$$j^{-,1}(x;z) = \begin{pmatrix} e^{-izx} \\ 0 \end{pmatrix} + o(1) \text{ as } x \to -\infty,$$

and

$$j^{-,2}(x;z) = \begin{pmatrix} 0 \\ e^{izx} \end{pmatrix} + o(1) \text{ as } x \to -\infty.$$

These two solutions are linearly independent and form a 2×2 fundamental solution matrix (i.e. the matrix whose columns are independent solution vectors):

$$J^-(x; z) = (j^{-,1}(x; z), j^{-,2}(x; z)).$$

with normalization condition

$$\lim_{x \to -\infty} J^-(x; z) e^{izx\sigma_3} = \mathbb{I}, \; \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Indeed, we have $\det(J^-(x; z)) = 1 \neq 0$ for all $x \in \mathbb{R}$ since

$$\frac{d}{dx} \det(J^-(x; z)) = 0 \text{ by (1.22) and } \det(J^-(x; z)) \to 1 \text{ as } x \to -\infty.$$

Similarly we can define Jost solutions $j^{+,1}(x; z), j^{+,2}(x; z)$ that are normalized in the limit $x \to +\infty$ and the associated normalized fundamental solution matrix

$$J^+(x; z) = (j^{+,1}(x; z), j^{+,2}(x; z)), \text{ with } \lim_{x \to +\infty} J^+(x; z) e^{izx\sigma_3} = \mathbb{I}.$$

Scattering matrix

The two fundamental solution matrices $J^\pm(x; z)$ both satisfy the 2×2 system (1.22):

$$(J^\pm)_x = \begin{pmatrix} -iz & u \\ \bar{u} & iz \end{pmatrix} J^\pm.$$

The columns of both matrices span the complete solution space and hence $j^{+,1}, j^{+,2}$ can be expressed as the linear combination of $j^{-,1}, j^{-,2}$ and vice versa. That is, there exists a matrix $S = S(z)$ such that

$$\begin{aligned} &J^+(x; z) = J^-(x; z) S(z), \quad \det(S(z)) = 1, \quad z \in \mathbb{R}, \\ &\text{i.e. } j^{+,1}(x; z) = s_{11}(z) j^{-,1}(x; z) + s_{21}(z) j^{-,2}(x; z), \\ &\text{and } j^{+,2}(x; z) = s_{12}(z) j^{-,1}(x; z) + s_{22}(z) j^{-,2}(x; z). \end{aligned}$$

This matrix $S(z)$ is the so-called scattering matrix.

Notice the symmetry in the system (1.22): If $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ solves (1.22), then

$$\tilde{\psi} := \begin{pmatrix} \overline{\psi_2} \\ \overline{\psi_1} \end{pmatrix} = \sigma_1 \overline{\psi}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

(with $\bar{f}$ denoting the complex conjugate of f) is also a solution of (1.22). Hence the jost solution $\widetilde{j^{-,1}}(x;z)$ also solves (1.22) with the asymptotic $\begin{pmatrix} 0 \\ e^{izx} \end{pmatrix}$ as $x \to -\infty$. By the unique solvability of the initial value problem (1.22)-(1.23) we know $j^{-,2}(x;z) = \widetilde{j^{-,1}}(x;z) = \sigma_1 \overline{j^{-,1}}(x;z)$. Therefore

$$\overline{J^{\pm}}(x;z) = (\overline{j^{\pm,1}}, \overline{j^{\pm,2}}) = \sigma_1(j^{\pm,2}, j^{\pm,1}) = \sigma_1 J^{\pm}(x;z)\sigma_1, \quad z \in \mathbb{R}.$$

Hence the scattering matrix $S(z) = J^-(x;z)^{-1}J^+(x;z)$ also satisfies

$$\overline{S}(z) = \sigma_1 S(z)\sigma_1, \quad z \in \mathbb{R}.$$

This, together with $\det(S(z)) = 1$ ensures that there exist two complex-valued functions $a(z), b(z)$ such that

$$S(z) = \begin{pmatrix} \overline{a(z)} & -\overline{b(z)} \\ -b(z) & a(z) \end{pmatrix}, \quad |a(z)|^2 - |b(z)|^2 = 1, \quad z \in \mathbb{R}. \tag{1.24}$$

The inverse scattering matrix $S(z)^{-1} = \begin{pmatrix} a(z) & \overline{b(z)} \\ b(z) & \overline{a(z)} \end{pmatrix}$.

The quantity $T(z) = 1/a(z)$ is called the transmission coefficient and the quantity $R(z) = b(z)/a(z)$ is called the reflection coefficient. Here is some explanation: We know from $J^-(x;z) = J^+(x;z)S(z)^{-1}$ that $j^{-,1}(x;z) = a(z)j^{+,1}(x;z) + b(z)j^{+,2}(x;z)$ and hence the solution $a(z)^{-1}j^{-,1}(x;z)$ has the following asymptotics

$$\begin{aligned} &T(z)\begin{pmatrix} e^{-izx} \\ 0 \end{pmatrix} \quad \text{as } x \to -\infty, \\ &\begin{pmatrix} e^{-izx} \\ 0 \end{pmatrix} + R(z)\begin{pmatrix} 0 \\ e^{izx} \end{pmatrix} \quad \text{as } x \to +\infty, \quad z \in \mathbb{R}. \end{aligned}$$

If we take e^{-izx} as the incoming wave from the right to the left, then after the disturbance modelled by the potential $u(x)$ there is a reflected wave of complex amplitude $R(z)$ and a transmitted wave of complex amplitude $T(z)$.

It is obvious to see that if $u \equiv 0$ then $J^-(x;z) = J^+(x;z) = e^{-izx\sigma_3}$ and hence $S(z) = \mathbb{I}$, $a(z) = 1$, $b(z) = 0$, $T(z) = 1$, $R(z) = 0$.

The direct transform

The direct transform for the defocusing NLS is a nonlinear mapping

$$(u \in L^1(\mathbb{R};\mathbb{C})) \mapsto (R : \mathbb{R} \mapsto \mathbb{C}).$$

The reflection coefficient $R(z)$ can be viewed as a nonlinear analogue of the Fourier transform of $u(x)$. Beals-Coifman (1984') showed that if $u \in \mathcal{S}(\mathbb{R})$ (the Schwartz space), then also $R \in \mathcal{S}(\mathbb{R})$ and $\sup_{z\in\mathbb{R}}|R(z)| \leqslant 1$. It is also interesting to calculate that if $u = \varepsilon \mathbf{1}_{[-L,L]}$, then $R(z) = \hat{u}(z) + O(\varepsilon^2)$ as $\varepsilon \to 0$ if we define $\hat{u}(z) = \int_{\mathbb{R}} e^{-2izx}u(x)dx = \varepsilon \sin(2zL)/z$.

1.2.2 Inverse scattering transform

Take the time t into account and view $z \in \mathbb{R}$ as parameter. Let $u = u(t,x)$ in the system (1.22) satisfy the defocusing cubic NLS (1.14) and $|u(t,\cdot)|$ decays to zero sufficiently fast for all $t \in \mathbb{R}$. Then the reflection coefficient $R(z) = R(t,z)$ also depends on t and we will see that $R(z;t)$ evolves in time (almost) in the same way as does the evolutionary $\hat{u}(t,\xi)$ for the linear Schrödigner equation (1.21).

We will prove (**Exercise**.) that the matrices

$$W^{\pm}(t,x;z) = J^{\pm}(t,x;z)e^{-2iz^2t\sigma_3}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are simultaneous fundamental solutions of the compatible linear systems (1.15)-(1.16). Hence we deduce

$$\partial_t J^{\pm}(t,x;z) = V(t,x;z)J^{\pm}(t,x;z) + 2iz^2 J^{\pm}(t,x;z)\sigma_3.$$

Therefore

$$\begin{aligned} \partial_t(S(t;z)) &= \partial_t(J^-(t,x;z)^{-1}J^+(t,x;z)) \\ &= -J^-(t,x;z)^{-1}\partial_t J^-(t,x;z)S(t;z) + J^-(t,x;z)^{-1}\partial_t J^+(t,x;z) \\ &= 2iz^2[S(t;z),\sigma_3], \end{aligned}$$

that is,

$$\begin{pmatrix} \partial_t\overline{a(t;z)} & -\partial_t\overline{b(t;z)} \\ -\partial_t b(t;z) & \partial_t a(t;z) \end{pmatrix} = 2iz^2 \begin{pmatrix} 0 & 2\overline{b(t;z)} \\ -2b(t;z) & 0 \end{pmatrix}.$$

We finally arrive at the time evolution of the transmission and reflection coefficients

$$\begin{aligned} &a(t;z) = a(0;z), \quad b(t;z) = e^{4iz^2t}b(0;z), \\ &\text{i.e.} T(t;z) = T(0;z) \text{ independent of the time,} \\ &\text{and } R(t;z) = e^{4iz^2t}R(0;z) \text{ evolves similarly as (1.21).} \end{aligned} \tag{1.25}$$

Inverse scattering transform

We give the following theorem without proof, which offers a way to recover the solution of the Cauchy problem (NLS) with $\kappa = 1$ from the associated Riemann-Hilbert problem:

Theorem 1.2. *Let $u_0 \in L^1(\mathbb{R};\mathbb{C})$ with $R_0 : \mathbb{R} \mapsto \mathbb{C}$ as the initial reflection coefficient. Then the solution of the Cauchy problem for the cubic nonlinear Schrödinger equation* (1.14) *with $\kappa = 1$ is*

$$u(t,x) = 2i \lim_{z\to\infty} zM_{12}(z;t,x),$$

where the matrix $M(z;t,x)$ is the solution of the following Riemann-Hilbert problem: Find a 2×2 matrix $M(z;t,x)$ such that

- *Analyticity - $M(z;t,x)$ is analytic of z for $z \in \mathbb{C}\setminus\mathbb{R}$;*
- *Jump Condition - The continuous boundary values $M_\pm(z;t,x)$ (from up and below respectively) on the real line $z \in \mathbb{R}$ are related by*

$$M_+(z;t,x) = M_-(z;t,x) \begin{pmatrix} 1-|R_0(z)|^2 & -e^{-2izx-4iz^2t}\overline{R_0(z)} \\ e^{2izx+4iz^2t}R_0(z) & 1 \end{pmatrix};$$

- *Normalization - $\lim_{z\to\infty} M(z;t,x) = \mathbb{I}$.*

Therefore we can solve the Cauchy problem (1.14) in the defocusing completely integrable case in the following way:

$$\begin{array}{ccc} u_0(x) & \text{- - - - - - - -} & u(t,x) \\ \text{direct scattering transform}\ \Big\downarrow & & \Big\uparrow\ \text{inverse scattering transform} \\ R_0(z) & \xrightarrow[\cdot e^{4iz^2t}]{} & R(t,z) \end{array}$$

However, the inverse scattering transform step is rather involved and it is hard to say that this machinery can work easier than other methods. Nevertheless it offers an algorithm to solve (NLS) and we can derive much information from the formulation itself, e.g. asymptotic behaviors of the solutions.

[26.10.2017]

[31.10.2017]

1.2.3 Transmission coefficient and reflection coefficient

As we have seen in Subsection 1.2.1 that the Jost solution $j^{-,1}$ of the ODE system (1.15) behaves asymptotically at infinity as follows:

$$\begin{pmatrix} e^{-izx} \\ 0 \end{pmatrix} \text{ as } x \to -\infty,$$
$$T^{-1}(z)\begin{pmatrix} e^{-izx} \\ 0 \end{pmatrix} + T^{-1}(z)R(z)\begin{pmatrix} 0 \\ e^{izx} \end{pmatrix} \text{ as } x \to +\infty,$$

the renormalised Jost solution $l^{-,1} = e^{izx} j^{-,1}$ solves (1.15) with the following asymptotic behaviour at infinity:

$$\begin{pmatrix}1\\0\end{pmatrix} \text{ as } x \to -\infty,$$
$$T^{-1}(z)\begin{pmatrix}1\\0\end{pmatrix} + T^{-1}(z)R(z)\begin{pmatrix}0\\e^{2izx}\end{pmatrix} \text{ as } x \to +\infty.$$

We derive from the explicit expression of $l^{-,1}$ in (A.27) in the appendix (if $u \in L^1(\mathbb{R};\mathbb{C})$) that

$$T^{-1} = \lim_{x\to+\infty}(l^{-,1})^1(x) = 1 + \int_{x_1<x_2} e^{2iz(x_2-x_1)}u(x_2)\bar{u}(x_1)\,\mathrm{d}x_1\,\mathrm{d}x_2$$
$$+ \text{ quartic term in } \{u,\bar{u}\} + \text{hexic term in } \{u,\bar{u}\} + \cdots$$

and

$$T^{-1}R = \lim_{x\to+\infty} e^{-2izx}(l^{-,1})^2(x) = \int_{\mathbb{R}} e^{-2izx_1}\bar{u}(x_1)\,\mathrm{d}x_1$$
$$+ \int_{\mathbb{R}} e^{-2izx_3}\bar{u}(x_3)\int_{x_1<x_2<x_3} e^{2iz(x_2-x_1)}u(x_2)\bar{u}(x_1)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3$$
$$+ \text{ quintic term in } \{u,\bar{u}\} + \text{septic term in } \{u,\bar{u}\} + \cdots$$

Formally if $\|u\| \ll 1$, then

$$\ln T^{-1}(z) \sim \int_{x_1<x_2} e^{2iz(x_2-x_1)}u(x_2)\bar{u}(x_1)\,\mathrm{d}x_1\,\mathrm{d}x_2,$$

and

$$R(z) \sim \int_{\mathbb{R}} e^{-2izx}\bar{u}(x)\,\mathrm{d}x = \sqrt{2\pi}\hat{\bar{u}}(2z),$$

such that we can view the Fourier transform as the "linear" part of the direct scattering transform $u(x) \mapsto R(z)$. We remark that although the dependence of R on u is nonlinear, we derive R from u by solving a *linear* ODE system (1.15).

<u>Invariant transmission coefficient</u>

We can extend the Jost solution $j^{-,1}(x;z)$ to the closed upper half plane, with the asymptotics

$$\begin{pmatrix}e^{-izx}\\0\end{pmatrix} + o(1)e^{\operatorname{Im} zx} \text{ as } x \to -\infty, \quad \begin{pmatrix}T^{-1}e^{-izx}\\0\end{pmatrix} + o(1)e^{\operatorname{Im} zx} \text{ as } x \to +\infty.$$

We will focus on the invariant "transmission coefficient" $T^{-1}(z)$ which is a well-defined holomorphic function on the closed upper half plane if $u(x) \in \mathcal{S}$. In the defocusing case, it also provides a holomorphic extension of $T(z)$ with $|T| \leqslant 1$, since there are no zeros of $T^{-1}(z)$ (corresponding to eigenvalues of the self-adjoint Lax operator) and $|T| \leqslant 1$ on $\mathbb{R}$, $T \to 1$ at infinity.

If $u(x) \in \mathcal{S}$, then we can expand $\ln T^{-1}(z)$ as $z \to \infty$:

$$\ln T^{-1}(z) = iM(2z)^{-1} - iP(2z)^{-2} + iE(2z)^{-3} + i\sum_{k=4}^{\infty} H_{k-1}(2z)^{-k}$$

where $M = M(u), P = P(u), E = E(u)$ are the conserved mass, momentum, energy defined in (1.8)-(1.9)-(1.10) respectively. By the conservation of $T(z)$, all the coefficients H_k in the above expansion are conserved by the cubic NLS flow and hence we derived infinite many conservation laws from the invariant transmission coefficient $T(z)$.

Roughly speaking, $M(u)$, $E(u)$ correspond to the L^2, H^1 regularity of the solution u, then whether or not the $2n$-th coefficient H_{2n} correspond to its H^n regularity? We have no idea about it. Nevertheless [Koch-Tataru 2016 arXiv] (see also [Killip-Visan-Zhang 2017 arXiv]) succeeded in reformulating the new conserved energies $E_s(u)$ from $\ln T^{-1}(z)$, which correspond to H^s-norms of the solution of the one dimensional cubic nonlinear focusing/defocusing Schrödinger equation, for all $s > -\frac{1}{2}$. We point out that the invariant "transmission coefficient" $T^{-1}(z)$ is a well-defined holomorphic function on the open upper half plane (not necessarily on the real axis) if $u \in H^s(\mathbb{R})$, $s > -\frac{1}{2}$.

The idea is as follows. If $\operatorname{Im} z > 0$, $u \in \mathcal{S}$ and $\|u\|_{H^s} \ll 1$, $s > -\frac{1}{2}$, then the leading term in $\ln T^{-1}(z)$ is $T_2(z) := \int_{x_1 < x_2} e^{2iz(x_2 - x_1)} u(x_2)\bar{u}(x_1)\,\mathrm{d}x_1\,\mathrm{d}x_2$. We can make use of Fourier transform $\hat{u}(\xi) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-i\xi x} u(x)\,\mathrm{d}x$ to rewrite T_2 as (**Exercise**)

$$T_2(z) = \int_{\mathbb{R}} \frac{i}{2z+\xi}\hat{u}(\xi)\bar{\hat{u}}(\xi)d\xi,$$

and hence

$$\frac{1}{\pi}\operatorname{Re} T_2(\frac{z}{2}) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{\operatorname{Im} z}{|z-\xi|^2}|\hat{u}(\xi)|^2\,\mathrm{d}\xi\,,$$

which is the harmonic function on the upper half plane with the trace $|\hat{u}(\xi)|^2$ on the real axis. By use of the trace formula, if $s \in (-\frac{1}{2}, 0)$, then we can rewrite the Sobolev norm $\|u\|_{H^s}^2 = \int_{\mathbb{R}}(1+|\xi|^2)^s|\hat{u}(\xi)|^2\,\mathrm{d}x$ (i.e. an integral on the real axis) in terms of T_2 as follows (i.e. an integral on the imaginary axis):

$$\|u\|_{H^s}^2 = -\frac{2\sin(\pi s)}{\pi}\int_1^\infty (\tau^2-1)^s \mathrm{Re}\, T_2(\frac{i\tau}{2})d\tau, \quad -\frac{1}{2} < s < 0.$$

Therefore in order to show the global-in-time boundedness of the Sobolev norm $\|u\|_{H^s}$ of the solution u to cubic (NLS) when $\|u_0\|_{H^s} \ll 1$, it suffices to verify that the integral of the remainder term $\int_1^\infty (\tau^2-1)^s \mathrm{Re}\,(\ln T^{-1} - T_2)(\frac{i\tau}{2})d\tau$ is much smaller than $\|u_0\|_{H^s}^2$ (indeed of order $\|u_0\|_{H^s}^4$). For general initial data u_0 which are not necessarily small, we use the symmetry property to get a global-in-time bound (depending nonlinearly on $\|u_0\|_{H^s}$) for the solution.

[31.10.2017]

[02.11.2018]

1.2.4 KdV case

The Korteweg-de Vries equation $\partial_t u + u_{xxx} - 6uu_x = 0$ (with the unknown function $u : \mathbb{R}\times\mathbb{R} \mapsto \mathbb{R}$) is also a completely integrable system and the associated Lax pair (i.e. the pair of operators L, P such that $L_t = [P, L]$) for KdV is given by

$$L_{\mathrm{KdV}} = -\partial_x^2 + u, \quad P_{\mathrm{KdV}} = -4\partial_x^3 + 3(u\partial_x + \partial_x u).$$

The scattering transform associated to KdV is defined via the spectral problem of the self-adjoint operator L_{KdV} :

$$L_{\mathrm{KdV}}\,\varphi = z^2\varphi.$$

L_{KdV} has continuous spectrum $\mathbb{R}^+$ and may have isolated (but possibly infinite) negative eigenvalues together with a resonance at frequency 0.

We can rewrite $L_{\mathrm{KdV}}\,\psi = z^2\psi$ in the form of the ODE system (1.15):

$$\psi_x = \begin{pmatrix} -iz & 1 \\ u & iz \end{pmatrix} \psi := U_{\mathrm{KdV}}\,\psi, \tag{1.26}$$

where $\psi = (\varphi, \varphi_x - iz\varphi)^T$. Then for $z \in \mathbb{R}$, we consider the Jost solution $j^{-,1}$ with the following asymptotics at infinity:

$$\begin{pmatrix} e^{-izx} \\ 0 \end{pmatrix} \text{ as } x \to -\infty,$$
$$\begin{pmatrix} T^{-1}(z)e^{-izx} - \frac{i}{2z}T^{-1}(z)R(z)e^{izx} \\ T^{-1}(z)R(z)e^{izx} \end{pmatrix} \text{ as } x \to +\infty,$$

and similarly the Jost solutions $j^{-,2}$, $j^{+,1}$, $j^{+,2}$. If u satisfies KdV equation, then $T_t = 0$ and $R_t = 8iz^3 R$.

If there are eigenvalues $z_n^2 < 0$ with $\operatorname{Im} z_n > 0$ (a different situation from defocusing cubic NLS where there are no eigenvalues for the Lax operator), then $a(z_n) = T^{-1}(z_n) = 0$ (i.e. z_n are poles of $T(z)$) and $j^{-,1}(x; z_n) = \tilde{C}_n j^{+,2}(x; z_n)$ for some constant $\tilde{C}_n = b(z_n)$. We define the normalizing coefficients $C_n = \frac{b(z_n)}{a'(z_n)}$ and the scattering data are $\{(z_n)_{n=1}^N, (C_n)_{n=1}^N, R(z)\}$. If u solves KdV equation, then $C_n(t) = C_n(0)e^{8iz_n^3 t} = C_n(0)e^{8(\operatorname{Im} z_n)^3 t}$ where the poles z_n of T are invariant under time evolution.

A few words on the focusing cubic NLS case. The focusing cubic NLS is also completely integrable, with the ODE system (1.15) replaced by

$$\psi_x = \begin{pmatrix} -iz & u \\ -\bar{u} & iz \end{pmatrix} \psi.$$

Then the corresponding *non self-adjoint* Lax operator may have isolated infinitely many eigenvalues on the upper half plane which may accumulate on the real axis even for Schwartz potentials u. Consider the specific initial data such that $R_0(z) = 0$ and $N = 1$ (only one discrete eigenvalue for the Lax operator), then the solution of the focusing cubic NLS reads explicitly as

$$2\tau e^{4i\tau^2 t} e^{i(2\xi x - 4\xi^2 t)} e^{i(\phi_1 + \pi/2)} \operatorname{sech}(2\tau(x - 4\xi t) - x_1),$$

where $z_1 = \xi + i\tau$, $c_1^2 = -iC_1(0)$, $c_1 = |c_1|e^{i\phi_1}$, $x_1 = \ln|c_1|/2\tau$. This is a soliton solution (1.13) (keeping in mind (1.12) and that we make the change of variable $u \mapsto \frac{1}{\sqrt{2}}u$ to derive the focusing cubic NLS $i\partial_t u + \partial_{xx} u = -2|u|^2 u$ here), with the velocity 4ξ and the amplitude 2τ.

A little history. [Zabusky & Kruskal 1965] discovered first numerically the solitary wave solution to the KdV equation, which interacted "elastically" with another such solution. Shortly after this discovery, the pioneers [Gardner-Greene-Kruskal-Miura 1967] related the resolution of KdV to the direct and inverse scattering transforms. [Lax 1968] considerably generalized these ideas. [Zakharov & Shabat 1972] showed that the inverse scattering method also worked for the nonlinear Schrödinger equation. [Ablowitz-Kaup-Newell-Segur 1973] developed a method to find a large class of evolutionary equations solvable by these techniques.

1.3 Other related models

1.3.1 Madelung transform

Performing the so-called Madelung transform $u(t, \sqrt{2}x) = \sqrt{\rho(t,x)}e^{i\phi(t,x)}$ (this is possible if u is away from zero), we obtain the following system for the unknown (ρ, v) with $v = \nabla\phi$ from (NLS):

$$\begin{cases} \partial_t \rho + \operatorname{div}_x(\rho v) = 0, \\ \partial_t v + v \cdot \nabla_x v + \nabla_x(\kappa \rho^{\frac{p-1}{2}}) = \nabla_x\big(\frac{\Delta\sqrt{\rho}}{2\sqrt{\rho}}\big). \end{cases} \tag{1.27}$$

(**Exercise**.) The above system is referred to as the hydrodynamic form of (NLS) because of its similarity of the compressible Euler system. The ρ-equation can be viewed as the continuity equation and the v-equation can be viewed as the momentum equation with an additional *quantum pressure* $\nabla_x\big(\frac{\Delta\sqrt{\rho}}{2\sqrt{\rho}}\big)$. The system (1.27) is also called quantum Euler equation or Euler-Korteweg equation in the theories of quantum fluids and Korteweg fluids.

1.3.2 The NLS hierarchy

Let $d = 1$. Recall the conserved mass M, momentum P, energy E and the Hamiltonian formulation of (NLS) in Subsection 1.1.4.

Consider the following Hamiltonians in the NLS hierarchy (see Appendix B)

$$\begin{aligned} H_0 &= \int_{\mathbb{R}} |u|^2 \,\mathrm{d}x\,, \\ H_1 &= \frac{1}{2}\frac{1}{i}\int_{\mathbb{R}} u\partial_x\bar{u}\,\mathrm{d}x \;=\frac{1}{2}\mathrm{Im}\int_{\mathbb{R}} u\partial_x\bar{u}\,\mathrm{d}x\,, \\ H_2 &= \frac{1}{4}\int_{\mathbb{R}} \big(|\partial_x u|^2 + |u|^4\big)\,\mathrm{d}x\,, \\ H_3 &= \frac{1}{8}\mathrm{Im}\int_{\mathbb{R}} \big(\partial_x u\partial_{xx}\bar{u} + 3|u|^2 u\partial_x\bar{u}\big)\,\mathrm{d}x\,, \quad \cdots \end{aligned}$$

The even ones are even with respect to complex conjugation and have a positive definite principal part, which are referred to as energies. The odd ones are odd if we replace u by $\bar{u}$, which are referred to as momenta. With respect to the symplectic form $\omega(f,g) = \mathrm{Im}\int_{\mathbb{R}} f\bar{g}\,\mathrm{d}x$, these Hamiltonians generate the corresponding Hamiltonian flows as follows:

$$\begin{aligned} &H_0 \text{ generates the phase shifts} : \partial_\theta u = \nabla_\omega H_0(u) = -2iu \Rightarrow u = e^{-2i\theta}u_0; \\ &H_1 \text{ generates the group of translations} : u(x+a) = \big[e^{a\nabla_\omega H_1}u\big](x); \\ &H_2 \text{ generates the defocusing cubic NLS flow} : 2i\partial_t u = -\partial_{xx}u + 2|u|^2 u; \\ &H_3 \text{ generates the defocusing mKdV flow} : 4\partial_t u = -\partial_{xxx}u + 6|u|^2\partial_x u, \cdots \end{aligned}$$

These Hamiltonian flows are commuting and $\{H_j, H_k\} = 0$.

1.3.3 Maxwell equation

Vacuum case

The propagation of an electromagnetic wave (e.g. a laser pulse) in vacuum is governed by Maxwell's equations:

$$\operatorname{curl}\mathcal{E} = -\partial_t\mathcal{B}, \quad \operatorname{curl}\mathcal{H} = \partial_t\mathcal{D}, \quad \operatorname{div}\mathcal{D} = 0, \quad \operatorname{div}\mathcal{B} = 0, \tag{1.28}$$

where $t \in \mathbb{R}$, $x \in \mathbb{R}^3$ are the time and space variables, $\mathcal{E}, \mathcal{H} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ are the electric and the magnetic fields respectively, and $\mathcal{D}, \mathcal{B} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ are the electric and magnetic induction fields respectively. In vacuum, $\mathcal{D}, \mathcal{B}$ are related to $\mathcal{E}, \mathcal{H}$ by the constitutive relations

$$\mathcal{D} = \epsilon_0\mathcal{E}, \quad \mathcal{B} = \mu_0\mathcal{H},$$

where ϵ_0, μ_0 are vacuum permittivity and permeability respectively.

Then we derive the wave equation for the electric field:

$$\partial_t^2\mathcal{E}_j - c^2\Delta\mathcal{E}_j = 0, \quad j = 1, 2, 3,$$

where $c = (\epsilon_0\mu_0)^{-\frac{1}{2}}$ is the speed of light in vacuum. (**Exercise**.) We look for the time-harmonic solution of the form

$\mathcal{E}_j(t, x) = e^{-i\omega_0 t}E(x) +$ c.c., with "c.c." standing for the complex conjugate,

where $E(x)$ satisfies the *scalar linear Helmholtz equation*

$$\Delta E + k_0^2 E = 0, \text{ with } k_0^2 = \frac{\omega_0^2}{c^2}.$$

We look for solutions of the Helmholtz equation of the form

$$E(x) = e^{ik_0x_3}\psi(x),$$

where ψ is the electric-field envelope (or amplitude) and satisfies

$$\partial_{x_3x_3}\psi + 2ik_0\partial_{x_3}\psi + (\partial_{x_1x_1} + \partial_{x_2x_2})\psi = 0.$$

In the *paraxial approximation* (i.e. ψ is slowly-varying in x_3-direction, compared with the carrier oscillation $e^{ik_0x_3}$), we can neglect $\partial_{x_3x_3}\psi$[1] (mathematically questionable) and the above equation for ψ becomes the *linear*

[1] Consider the plane waves $E = E_c e^{ik_1x_1+ik_2x_2+ik_3x_3}$ with $k_1^2+k_2^2+k_3^2 = k_0^2$ for the scalar linear Helmholtz equation, such that $\psi = E_c e^{ik_1x_1+ik_2x_2+i(k_3-k_0)x_3}$. Then for paraxial plane waves $k_3 - k_0 \ll k_0$,

$$\frac{|\psi_{33}|}{|k_0\psi_3|} = \frac{(k_0-k_3)^2}{k_0|k_0-k_3|} = \frac{|k_0-k_3|}{k_0} \ll 1;$$
$$\frac{|\psi_{33}|}{|\psi_{11}+\psi_{22}|} = \frac{(k_0-k_3)^2}{|k_0^2-k_3^2|} = \frac{|k_0-k_3|}{(k_0+k_3)} \ll 1.$$

Schrödinger equation

$$2ik_0\partial_{x_3}\psi + (\partial_{x_1x_1} + \partial_{x_2x_2})\psi = 0.$$

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[09.11.2018]

Dielectric medium

When an electric field applies on a dielectric medium, it induces an additional electric field, which is called the *polarization field*. Hence the electrical induction field $\mathcal{D}$ in a dielectric medium becomes the sum of the original electric field and the polarization field:

$$\mathcal{D} = \epsilon_0\mathcal{E} + \mathcal{P}.$$

For simplicity let us assume that $\mathcal{E}, \mathcal{D}, \mathcal{P} \in \mathbb{R}$ (that is, the electric field is linearly polarized: $(\mathcal{E}, 0, 0)$). At low intensities, the dependence of $\mathcal{P}$ on $\mathcal{E}$ is linear:

$$\mathcal{P} = \mathcal{P}_{\text{lin}} = \epsilon_0\chi^{(1)}(\omega_0)\mathcal{E},$$

where $\chi^{(1)}$ is the *first-order optical susceptibility.* Hence the scalar Helmholtz equation in a linear dielectric becomes

$$\Delta E + k_0^2 E = 0, \qquad \text{with } k_0^2 = \frac{\omega_0^2}{c^2}n_0^2, \tag{1.29}$$

where $n_0 = \sqrt{1 + \chi^{(1)}}$ is the (linear) *index of refraction of the medium.*

As $\mathcal{E}$ increases, the dependence becomes nonlinear and in the *weakly nonlinear regime* we have

$$\mathcal{P} = \mathcal{P}_{\text{lin}} + \mathcal{P}_{\text{nl}}, \quad \mathcal{P}_{\text{nl}} \approx \chi^{(3)}(\omega_0)\mathcal{E}^3 \ll \mathcal{P}_{\text{lin}},$$

where $\chi^{(2j)} = 0$ in isotropic materials. Let $\mathcal{E} = e^{-i\omega_0 t}E + c.c.$, then (we neglect the part with frequency $3\omega_0$)

$$\mathcal{P}_{\text{nl}} \approx \chi^{(3)}\big(3|E|^2 E e^{-i\omega_0 t} + c.c.\big) = 3\chi^{(3)}|E|^2\mathcal{E} := 4\epsilon_0 n_0 n_2 |E|^2\mathcal{E},$$

where we defined the *Kerr coefficient*

$$n_2 = \frac{3\chi^{(3)}}{4\epsilon_0 n_0}.$$

Therefore,

$$\mathcal{D} = \epsilon_0 n^2\mathcal{E}, \text{ with } n^2 = n_0^2\big(1 + \frac{4n_2}{n_0}|E|^2\big).$$

Hence if we *formally* replace n_0^2 by n^2 in the scalar nonlinear Helmholtz equation (1.29), we get the *scalar nonlinear Helmholtz equation* for the propagation of a linearly-polarized laser beam in a Kerr medium ($\mathcal{E}$ does not necessarily satisfy the wave equation $n^2\partial_t^2\mathcal{E} - c^2\Delta\mathcal{E} = 0$)

$$\Delta E + k^2 E = 0, \qquad \text{with } k^2 = k_0^2\big(1 + \frac{4n_2}{n_0}|E|^2\big).$$

We substitute $E = e^{ik_0x_3}\psi$ into the above equation and apply the paraxial approximation ($\psi_{x_3x_3} \ll k_0\psi_{x_3}$), to derive the nonlinear Schrödinger equation (NLS) for ψ:

$$2ik_0\partial_{x_3}\psi + (\partial_{x_1x_1} + \partial_{x_2x_2})\psi + k_0^2\frac{4n_2}{n_0}|\psi|^2\psi = 0. \tag{1.30}$$

(**Exercise**: Use the symmetry property to rewrite the above equation into the standard form (NLS).)

To conclude, the above (NLS) is the leading order model for paraxial propagation of intense linearly-polarized continuous wave laser beams in a homogeneous Kerr medium, in which ψ is the slowly-varying amplitude of the electric field, x_3 is the direction of propagation.

1.3.4 Water waves

This subsection can be found in [Zakharov 1968]. We consider the potential flow of an ideal fluid in the domain $\{(x, y, z) \mid (x, y) \in \mathbb{R}^2,\ z \in (-\infty, \eta]\}$, in the presence of a gravity field. Let $\eta = \eta(x, y, t)$ be the shape of the surface of the fluid and let $\phi = \phi(x, y, z, t)$ be the hydrodynamic potential (that is, the velocity field $v = (\partial_x, \partial_y, \partial_z)^T\phi$). Denote simply $\nabla = \nabla_{x,y}$, $\Delta = \Delta_{x,y}$ on the xy-plane. Then the dynamic of the ideal fluid is governed by the Laplace's equation for ϕ:

$$\Delta\phi + \partial_{zz}\phi = 0, \quad (x, y) \in \mathbb{R}^2, \quad z \in (-\infty, \eta),$$

together with the boundary conditions at the surface (neglecting the surface tension):

$$\begin{aligned} &\text{kinetic b.c.: } \partial_t\eta = \partial_z|_{z=\eta}\phi - \nabla\eta\cdot\nabla\phi|_{z=\eta}, \\ &\text{dynamic b.c.: } \partial_t\phi|_{z=\eta} + g\eta = -\frac{1}{2}\big(|\nabla\phi|^2 + (\partial_z\phi)^2\big)|_{z=\eta}, \end{aligned} \tag{1.31}$$

where g is the gravity acceleration and the condition at infinity:

$$\phi \to 0, \text{ as } z \to -\infty.$$

We introduce $\psi = \psi(x,y,t) = \phi\big(x,y,z=\eta(x,y),t\big)$, then $\partial_t\psi = \partial_t\phi + \partial_t\eta\partial_z|_{z=\eta}\phi$. Hence the above boundary conditions (1.31) become the evolutionary equations for (η,ψ):

$$\begin{cases} \partial_t\eta = \partial_z|_{z=\eta}\phi - \nabla\eta\cdot\nabla\phi|_{z=\eta}, \\ \partial_t\psi + g\eta = -\frac{1}{2}|\nabla\phi|^2 + \frac{1}{2}(\partial_z\phi)^2|_{z=\eta} - \partial_z\phi(\nabla\eta\cdot\nabla\phi)|_{z=\eta}, \end{cases} \tag{1.32}$$

where ϕ solves the Laplace's equation in $\mathbb{R}^2\times(-\infty,\eta]$ together with the boundary conditions $\phi|_{z=\eta}=\psi$, $\phi\to 0$ as $|z|\to\infty$. We remark that the system (1.32) are Hamiltonian's equations:

$$\partial_t\eta = \frac{\delta H}{\delta\psi}, \quad \partial_t\psi = -\frac{\delta H}{\delta\eta}, \tag{1.33}$$

where H is the Hamiltonian

$$H = \frac{1}{2}\int_{\mathbb{R}^2}\int_{-\infty}^{\eta}\Big(|\nabla\phi|^2 + (\partial_z\phi)^2\Big)dz\,\mathrm{d}x\,\mathrm{d}y + \frac{1}{2}g\int_{\mathbb{R}^2}\eta^2\,\mathrm{d}x\,\mathrm{d}y\,,$$

and ψ is the generalized coordinate and η is the generalized momentum.

We take the Fourier transform of η,ψ on the xy-plane, and linearize (1.32) under the *small amplitude assumption* :

$$\begin{cases} \partial_t\hat\eta - |k|\hat\psi = 0, \\ \partial_t\hat\psi + g\hat\eta = 0, \end{cases} \tag{1.34}$$

where we used the expansion for ϕ: $\hat\phi(k) = e^{|k|z}\hat\psi(k) + O(|\eta|)$. Therefore we derive the dispersion relation

$$\omega = \omega(k) = \sqrt{g|k|}, \quad k = (k_x,k_y).$$

We introduce $a = a(k) = \frac{1}{\sqrt{2}}\big(\hat\eta(k)(\frac{\omega(k)}{|k|})^{\frac{1}{2}} + i\hat\psi(k)(\frac{|k|}{\omega(k)})^{\frac{1}{2}}\big)$ [2] to diagnize the linear equations (1.34):

$$\partial_t a(k) + i\omega(k)a(k) = 0.$$

Then Hamilton's equation (1.33) becomes a single equation $\partial_t a(k) = -i\frac{\delta H}{\delta\bar a(k)}$, where $H = \int_{\mathbb{R}^2}\omega(k)a(k)\bar a(k)dk + O(|a|^3)$, and we write (not obviously)

$$\begin{aligned}\partial_t a(k) + i\omega(k)a(k) = -i\int\Big(&V(-k,k_1,k_2)a(k_1)a(k_2)\delta(k-k_1-k_2) \\ &+ 2V(-k_1,k,k_2)\bar a(k_2)a(k_1)\delta(k-k_1+k_2) + V(k,k_1,k_2)\bar a(k_1)\bar a(k_2)\delta(k+k_1+k_2)\Big)dk_1dk_2 \\ &- i\int W(k,k_1,k_2,k_3)\bar a(k_1)a(k_2)a(k_3)\delta(k+k_1-k_2-k_3)dk_1dk_2dk_3 + O(|a|^4),\end{aligned}$$

[2]Then $\bar a(-k) = \frac{1}{\sqrt{2}}\big(\hat\eta(k)(\frac{\omega(k)}{|k|})^{\frac{1}{2}} - i\hat\psi(k)(\frac{|k|}{\omega(k)})^{\frac{1}{2}}\big)$.

where $V(k,k_1,k_2)$, $W(k,k_1,k_2,k_3)$ depend on their arguments explicitly (details omitted here) and δ is the delta function.

Under the small amplitude assumption we write $a(k)$ as

$$a(k) = \big(A(k,t) + f(k,t)\big)e^{-i\omega(k)t},$$

where $A(k,t)$ changes slowly in comparison with f while $|f| \ll |A| \ll 1$. Assuming A constant when f varies, we integrate the equation of a with respect to the time to arrive at the following expression for f (up to $|A|^3$):

$$\begin{aligned} f = -\int \Big(& V(-k,k_1,k_2)\frac{\exp it\big(\omega(k)-\omega(k_1)-\omega(k_2)\big)}{\omega(k)-\omega(k_1)-\omega(k_2)}A(k_1)A(k_2)\delta(k-k_1-k_2) \\ & + 2V(-k_1,k,k_2)\frac{\exp it\big(\omega(k)+\omega(k_1)-\omega(k_2)\big)}{\omega(k)+\omega(k_1)-\omega(k_2)}\bar{A}(k_2)A(k_1)\delta(k-k_1+k_2) \\ & + V(k,k_1,k_2)\frac{\exp it\big(\omega(k)+\omega(k_1)+\omega(k_2)\big)}{\omega(k)+\omega(k_1)+\omega(k_2)}\bar{A}(k_1)\bar{A}(k_2)\delta(k+k_1+k_2)\Big)dk_1dk_2. \end{aligned}$$

In the evolutionary equation for A we retain only the terms proportional to Af which contain the most slowly varying exponents, such that

$$\begin{aligned} \partial_t A = -i\int & T(k,k_1,k_2,k_3)\delta(k+k_1-k_2-k_3) \\ & \times \exp it\big(\omega(k)+\omega(k_1)-\omega(k_2)-\omega(k_3)\big)\bar{A}(k_1)A(k_2)A(k_3)dk_1dk_2dk_3, \end{aligned}$$

where T depends explicitly on its arguments (details omitted here).

Under the narrow wave packet assumption: $|\xi| = |k-k_0| \ll |k_0|$, we expand $\omega(k)$ in powers of $\xi = (\xi_1,\xi_2)$ (ξ_1, ξ_2 are the projections of the vector ξ along and perpendicular to the vector $k-k_0$ respectively) around k_0 as follows:

$$\begin{aligned} \omega(k) &= \omega(k_0) + \xi_1 c_g + \frac{1}{2}(\lambda_1\xi_1^2 + \lambda_2\xi_2^2) + \cdots, \\ c_g &= \partial_k|_{k=k_0}\omega, \quad \lambda_1 = \partial_k^2|_{k=k_0}\omega, \quad \lambda_2 = \frac{c_g}{k_0}. \end{aligned}$$

Finally we introduce

$$b(k,t) = A(k,t)e^{i(\xi_1 c_g + \frac{1}{2}(\chi_1\xi_1^2+\chi_2\xi_2^2))t},$$

and verify that its inverse Fourier transform $\check{b}(x_1,x_2,t)$ with respect to $\xi = (\xi_1,\xi_2)$ satisfies the cubic nonlinear Schrödinger equation (NLS):

$$\partial_t\check{b} + c_g\partial_1\check{b} - \frac{i}{2}\big(\lambda_1\partial_{11}\check{b} + \lambda_2\partial_{22}\check{b}\big) = -i\kappa|\check{b}|^2\check{b}. \tag{1.35}$$

2 Wellposedness

Definition 2.1 (LWP & GWP)**.** *The Cauchy problem* (NLS)

$$\begin{cases} i\partial_t u + \Delta u = \kappa |u|^{p-1}u, \\ u|_{t=0} = u_0(x), \end{cases}$$

is said to be locally well-posed LWP *in* $H^s(\mathbb{R}^d)$ *if for any initial data* $u_0 \in H^s(\mathbb{R}^d)$*, there exists a positive time* $T > 0$ *and a unique solution* $u \in C([-T,T]; H^s(\mathbb{R}^d))$ *of* (NLS) *such that there exists a neighbourhood* U *of* u_0 *in* $H^s(\mathbb{R}^d)$ *and the flow map*

$$\Phi : U \mapsto H^s(\mathbb{R}^d), \quad u_0 \mapsto u(t,\cdot)$$

is continuous for any $t \in (-T,T)$*.*

We say that (NLS) *is* globally well-posedness GWP *in* $H^s(\mathbb{R}^d)$ *if the above holds on any time interval* $[-T,T]$*,* $T > 0$*.*

Recall the famous Hadamard's example of the ill-posed Cauchy problem for the Laplace equation:

$$\begin{cases} v_{tt} + v_{xx} = 0, \\ v|_{t=0} = 0, \quad v_t|_{t=0} = f(x). \end{cases}$$

Exercise. Show the existence and the uniqueness results of the solution for the above Cauchy problem with the following initial data sequence

$$(v, v_t)|_{t=0} = (0, f_n) = (0, e^{-\sqrt{n}} n \sin(nx)) \to 0 \text{ as } n \to \infty, \text{ in any } C_b^k(\mathbb{R}),\ k \in \mathbb{N},$$

while the continuity of the flow map fails.

We will show the well-posedness results in the subcritical cases for (NLS) in this section. Recalling the Duhamel formula (Duhamel), we would like to apply the fixed point theorem to show the unique existence of the solution $u \in X_T \subset C([-T,T]; H^s(\mathbb{R}^d))$. The choice of the functional space X_T is crucial and we have to make sure that the linear map $u_0 \mapsto S(t)u_0$ is from $H^s(\mathbb{R}^d)$ to X_T while the (nonlinear) map $u \mapsto \int_0^t S(t-t')(f(u)(t'))\,dt'$, $f(u) = \kappa|u|^{p-1}u$ is from X_T to X_T. Finally we can choose the time T small enough such that these maps are contraction mappings and hence the fixed point theorem works. The mass/energy conservation laws then imply GWP in the L^2/H^1 framework respectively.

2.1 Strichartz estimates

Recall Proposition 1.1 that the Schrödinger group $S(t)$ maps $L^{r'}(\mathbb{R}^d)$ to $L^r(\mathbb{R}^d)$, $r \geqslant 2$ with the operator norm $|4\pi t|^{-(\frac{d}{2}-\frac{d}{r})}$. It is also obvious from the definition of $S(t)$: $\widehat{S(t)g}(\xi) = e^{-it|\xi|^2}\widehat{g}(\xi)$ and the definition of H^s-norm (1.5) that

$$\|S(t)g\|_{H^s(\mathbb{R}^d)} = \|g\|_{H^s(\mathbb{R}^d)}, \quad \forall s \in \mathbb{R}\,.$$

Remark 2.1. ***(Exercise.)*** *The operator $S(t)$, $t > 0$ does not map*

- *from $L^2(\mathbb{R}^d)$ to $L^r(\mathbb{R}^d)$ or from $L^r(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$ for $r \neq 2$;*
- *from $L^r(\mathbb{R}^d)$ to $L^r(\mathbb{R}^d)$ for $r \neq 2$;*
- *from $L^r(\mathbb{R}^d)$ to $L^{r_1}(\mathbb{R}^d)$ for any $r > 2$;*
- *from $H^s(\mathbb{R}^d)$ to $H^{s'}(\mathbb{R}^d)$ for $s' > s$.*

The heat semigroup $e^{t\Delta} = A_t *$ with $A_t = (4\pi t)^{-\frac{d}{2}} e^{-|\cdot|^2/4t} \in L^1(\mathbb{R}^d)$, $t > 0$ maps from L^r to L^r and from H^s to $H^{s'}$, for any $r \in [1, \infty]$ and $s' \geqslant s$.

We will show more estimates for $S(t)$ in this subsection, namely the well-known Strichartz estimates.

Theorem 2.1. *[Strichartz estimates for the Schrödinger semigroup] Let (q, r) be admissible exponent pair, i.e.*

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2}, \quad 2 \leqslant q, r \leqslant \infty, \quad (q, r, d) \neq (2, \infty, 2). \tag{2.36}$$

Then for any admissible exponent pairs $(q, r), (\widetilde{q}, \widetilde{r})$, we have the following homogeneous Strichartz estimate (for the solution $S(t)u_0$ of the homogeneous problem $i\partial_t u + \Delta u = 0, u|_{t=0} = u_0$)

$$\|S(t)u_0\|_{L^q_t L^r_x(\mathbb{R}\times\mathbb{R}^d)} \leqslant C(q, r, d)\|u_0\|_{L^2_x(\mathbb{R}^d)}, \tag{2.37}$$

and the inhomogeneous Strichartz estimate (for the solution $\int_0^t S(t-t')f(t')\,\mathrm{d}t'$ of the inhomogeneous problem $i\partial_t u + \Delta u = f, u|_{t=0} = 0$)

$$\Big\|\int_0^t S(t-t')f(t')\,\mathrm{d}t'\Big\|_{L^q_t L^r_x(\mathbb{R}\times\mathbb{R}^d)} \leqslant C(q, r, \widetilde{q}, \widetilde{r}, d)\|f\|_{L^{\widetilde{q}'}_t L^{\widetilde{r}'}_x(\mathbb{R}\times\mathbb{R}^d)}, \tag{2.38}$$

where $\frac{1}{q} + \frac{1}{q'} = 1$ and the space-time norm $\|\cdot\|_{L^q_t L^r_x(\mathbb{R}\times\mathbb{R}^d)}$ is defined to be

$$\|g\|_{L^q_t L^r_x(\mathbb{R}\times\mathbb{R}^d)} = \Big\|\|g(t,\cdot)\|_{L^r_x(\mathbb{R}^d)}\Big\|_{L^q_t(\mathbb{R})}.$$

In the above the time definition domain $\mathbb{R}$ can be replaced by any time interval $[-T, T]$, $T > 0$.

Remark 2.2.

- *By virtue of* (2.37)*, if* $u_0 \in L^2$*, then* $S(t)u_0 \in L^r$ *with*

$$2 \leqslant r \leqslant \infty \text{ if } d = 1, \quad 2 \leqslant r < \infty \text{ if } d = 2, \quad 2 \leqslant r \leqslant \frac{2d}{d-2} \text{ if } d \geqslant 3,$$

for almost all $t \in \mathbb{R}$ *and the norm* $\|S(t)u_0\|_{L^r}$*,* $r > 2$ *decays faster than* $|t|^{-\frac{1}{q}}$ *for almost all the time. This indicates the smooth and decay effects of* $S(t)$*. On the other side we know that there exists* $u_0 \in L^2$ *and* $t \in \mathbb{R}$ *such that* $S(t)u_0 \notin L^r$ *since the operator* $S(t)$ *is not a map from* L^2 *to* L^r*.*

- *There are infinite many admissible exponent pairs and we can always have the trivial case* $(q, r) = (\infty, 2)$ *and the particular case* $q = r = 2(d+2)/d$*. If* $d = 1$*, then* $q \geqslant 4$ *and*

$$\|S(t)u_0\|_{L^4_t L^\infty_x(\mathbb{R}\times\mathbb{R})} + \|S(t)u_0\|_{L^6_{t,x}(\mathbb{R}\times\mathbb{R})} \leqslant C\|u_0\|_{L^2(\mathbb{R})}.$$

- *The equality for the admissible exponent paris can be seen as follows: Let* u *be a solution of the homogeneous linear Schrödinger equation, then the rescaled solution* $u_\lambda(t,x) = u(\frac{t}{\lambda^2}, \frac{x}{\lambda})$ *with rescaled initial data* $u_{0,\lambda} = u(\frac{x}{\lambda})$*,* $\lambda > 0$ *such that*

$$\|u_\lambda\|_{L^q_t L^r_x(\mathbb{R}\times\mathbb{R}^d)} = \lambda^{(\frac{2}{q}+\frac{d}{r})}\|u\|_{L^q_t L^r_x(\mathbb{R}\times\mathbb{R}^d)}, \quad \|u_{0,\lambda}\|_{L^2_x(\mathbb{R}^d)} = \lambda^{\frac{d}{2}}\|u_0\|_{L^2_x(\mathbb{R}^d)},$$

also solves the linear Schrödinger equation. If (2.37) *holds for* u*, then it also holds for* u_λ *for any* $\lambda > 0$ *and the only possibility is* $\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$*.*

Proof. **Step 1 From $L^{q'}_t L^{r'}_x$ to $L^q_t L^r_x$ for $2 < q < \infty$**

By use of the estimate (1.4), we know for any $-\infty \leqslant t_1 < t_2 \leqslant \infty$

$$\begin{aligned}\Big\|\int_{t_1}^{t_2} S(t-t')f(t')\,\mathrm{d}t'\Big\|_{L^q_t L^r_x(\mathbb{R}\times\mathbb{R}^d)} &\leqslant \Big\|\int_{t_1}^{t_2} (4\pi|t-t'|)^{-(\frac{d}{2}-\frac{d}{r})}\|f(t')\|_{L^{r'}_x(\mathbb{R}^d)}\,\mathrm{d}t'\Big\|_{L^q_t(\mathbb{R})} \\ &\leqslant \Big\|\int_{\mathbb{R}} (4\pi|t-t'|)^{-\frac{2}{q}}\|f(t')\|_{L^{r'}_x(\mathbb{R}^d)}\,\mathrm{d}t'\Big\|_{L^q_t(\mathbb{R})}.\end{aligned}$$

By the Hardy-Littlewood-Sobolev inequality (**Exercise**)

$$\begin{aligned}&\|g * |\cdot|^{-\alpha}\|_{L^q(\mathbb{R}^n)} \leqslant C(p,q,\alpha,n)\|g\|_{L^m(\mathbb{R}^n)}, \\ &1 + \frac{1}{q} = \frac{1}{m} + \frac{\alpha}{n}, \quad 0 < \alpha < n, \quad 1 < m < q < \infty,\end{aligned}$$

with

$$n = 1, \quad \alpha = \frac{d}{2} - \frac{d}{r} = \frac{2}{q}, \quad m = q', \ (2 < q < \infty),$$

we derive from the above inequality that for any $-\infty \leqslant t_1 < t_2 \leqslant \infty$ ($t_1 < t_2$ may be any two functions of t)

$$\Big\| \int_{t_1}^{t_2} S(t-t') f(t') \, \mathrm{d}t' \Big\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^d)} \leqslant C(q,r,d) \|f\|_{L_t^{q'} L_x^{r'}(\mathbb{R}\times\mathbb{R}^d)}.$$

Step 2 From $L_t^{q'} L_x^{r'}$ to $L_t^\infty L_x^2$ for $2 < q < \infty$

We calculate for any $t \in \mathbb{R}$, $-\infty \leqslant t_1 < t_2 \leqslant \infty$

$$\begin{aligned}
&\Big\| \int_{t_1}^{t_2} S(t-t') f(t') \, \mathrm{d}t' \Big\|_{L_x^2(\mathbb{R}^d)}^2 = \Big\langle \int_{t_1}^{t_2} S(t-t') f(t') \, \mathrm{d}t' \, , \int_{t_1}^{t_2} S(t-t'') f(t'') \, dt'' \Big\rangle_{L_x^2(\mathbb{R}^d)} \\
&= \int_{t_1}^{t_2} \int_{t_1}^{t_2} \langle S(t-t') f(t'), S(t-t'') f(t'') \rangle_{L_x^2(\mathbb{R}^d)} \, dt' \, dt'' \\
&= \int_{t_1}^{t_2} \int_{t_1}^{t_2} \langle f(t'), S(t'-t'') f(t'') \rangle_{L_x^2(\mathbb{R}^d)} \, dt' \, dt'' \\
&= \int_{t_1}^{t_2} \Big\langle f(t'), \int_{t_1}^{t_2} S(t'-t'') f(t'') \, dt'' \Big\rangle_{L_x^2(\mathbb{R}^d)} dt' \\
&\leqslant \int_{\mathbb{R}} \|f(t')\|_{L_x^{r'}(\mathbb{R}^d)} \int_{\mathbb{R}} |4\pi(t'-t'')|^{-\frac{2}{q}} \|f(t'')\|_{L_x^{r'}(\mathbb{R}^d)} dt'' \, \mathrm{d}t' \ \text{ by (1.4)} \\
&\leqslant C \|f\|_{L_t^{q'} L_x^{r'}(\mathbb{R}\times\mathbb{R}^d)}^2 \ \text{ by Hardy-Littlewood-Sobolev inequality.}
\end{aligned}$$

Step 3 Proof of (2.37) and from $L_t^1 L_x^2$ to $L_t^q L_x^r$ by duality

By duality, we derive (2.37) by Step 2

$$\begin{aligned}
\|S(t)u_0\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^d)} &= \sup_{\|g\|_{L_t^{q'} L_x^{r'}(\mathbb{R}\times\mathbb{R}^d)} \leqslant 1} \Big| \int_{\mathbb{R}} \langle S(t)u_0, g(t) \rangle_{L_x^2(\mathbb{R}^d)} dt \Big| \\
&= \sup_{\|g\|_{L_t^{q'} L_x^{r'}(\mathbb{R}\times\mathbb{R}^d)} \leqslant 1} \Big| \int_{\mathbb{R}} \langle u_0, S(-t)g(t) \rangle_{L_x^2(\mathbb{R}^d)} dt \Big| \\
&\leqslant \sup_{\|g\|_{L_t^{q'} L_x^{r'}(\mathbb{R}\times\mathbb{R}^d)} \leqslant 1} \|u_0\|_{L_x^2(\mathbb{R}^d)} \Big\| \int_{\mathbb{R}} S(0-t') g(t') dt' \Big\|_{L^2} \\
&\leqslant C \|u_0\|_{L_x^2(\mathbb{R}^d)} \ \text{ by Step 2.}
\end{aligned}$$

Similarly, we can show for any $-\infty \leqslant t_1 < t_2 \leqslant \infty$,

$$
\begin{aligned}
&\Big\| \int_{t_1}^{t_2} S(t-t') f(t')\,\mathrm{d}t' \Big\|_{L^q_t L^r_x(\mathbb{R}\times\mathbb{R}^d)} \\
&= \sup_{\|g\|_{L^{q'}_t L^{r'}_x(\mathbb{R}\times\mathbb{R}^d)} \leqslant 1} \Big| \int_{\mathbb{R}} \Big\langle \int_{t_1}^{t_2} S(t-t') f(t')\,\mathrm{d}t'\,, g(t) \Big\rangle_{L^2_x(\mathbb{R}^d)} dt \Big| \\
&= \sup_{\|g\|_{L^{q'}_t L^{r'}_x(\mathbb{R}\times\mathbb{R}^d)} \leqslant 1} \Big| \int_{t_1}^{t_2} \Big\langle f(t'), \int_{\mathbb{R}} S(t'-t) g(t) dt \Big\rangle_{L^2_x(\mathbb{R}^d)} dt' \Big| \\
&\leqslant C \|f\|_{L^1_t L^2_x(\mathbb{R}\times\mathbb{R}^d)} \text{ by Step 2.}
\end{aligned}
$$

If $(t_1, t_2) = (0, t)$, then we just take the integral intervals $(0,\infty)$ and (t',∞) for the variables t' and t respectivly.

Step 4 Proof of (2.38) **by interpolation**

We have shown in Step 1 and Step 2 that the linear operator

$$
f \mapsto \int_0^t S(t-t') f(t') dt'
$$

is bounded from $L^{\tilde{q}'}_t L^{\tilde{r}'}_x$ to $L^{\tilde{q}}_t L^{\tilde{r}}_x$ and from $L^{\tilde{q}'}_t L^{\tilde{r}'}_x$ to $L^\infty_t L^2_x$. By the log-convexity of L^p-norms,

$$
\|g\|_{L^{p_\theta}} \leqslant \|g\|^{1-\theta}_{L^{p_0}} \|g\|^{\theta}_{L^{p_1}}, \text{ with } \frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1},
$$

the above operator is bounded from $L^{\tilde{q}'}_t L^{\tilde{r}'}_x$ to $L^q_t L^r_x$ if $2 < \tilde{q} \leqslant q \leqslant \infty$.

Similarly we have shown in Step 1 and Step 3 that the above linear operator is bounded from $L^{q'}_t L^{r'}_x$ to $L^q_t L^r_x$ and from $L^1_t L^2_x$ to $L^q_t L^r_x$ and hence from $L^{\tilde{q}'}_t L^{\tilde{r}'}_x$ to $L^q_t L^r_x$ if $1 \leqslant \tilde{q}' \leqslant q' < 2$.

These two cases complete the estimate (2.38) for $2 < q \leqslant \infty$.

Step 5 Endpoint case $q = 2, r = \frac{2d}{d-2}$ **for** $d \geqslant 3$: See [Keel-Tao 1998]. □

Remark 2.3 (TT^* argument)**.** *We can rewrite the proof in a more elegant way. Let* $T : L^2(\mathbb{R}^d) \mapsto L^q(\mathbb{R}; L^r(\mathbb{R}^d))$ *be defined as*

$$
(Tf)(t,x) = S(t) f(x),
$$

then its formal adjoint $T^* : L^{q'}(\mathbb{R}; L^{r'}(\mathbb{R}^d)) \mapsto L^2(\mathbb{R}^d)$ *is defined as*

$$
(T^* g)(x) = \int_{-\infty}^{\infty} S(-t) g(t,x) dt,
$$

and their composition $TT^* : L^{q'}(\mathbb{R}; L^{r'}(\mathbb{R}^d)) \mapsto L^q(\mathbb{R}; L^r(\mathbb{R}^d))$ *reads as*

$$(TT^* g)(t,x) = \int_{-\infty}^{\infty} S(t-t')g(t',x)dt'.$$

Then the following a priori estimates are equivalent:

$$\|Tf\|_{L^q(\mathbb{R};L^r(\mathbb{R}^d))} \lesssim \|f\|_{L^2(\mathbb{R}^d)},$$
$$\|T^* g\|_{L^2(\mathbb{R}^d)} \lesssim \|g\|_{L^{q'}(\mathbb{R};L^{r'}(\mathbb{R}^d))},$$
$$\|TT^* g\|_{L^q(\mathbb{R};L^r(\mathbb{R}^d))} \lesssim \|g\|_{L^{q'}(\mathbb{R};L^{r'}(\mathbb{R}^d))},$$

where the last inequality for (q,r) *admissible exponent pair with* $2 < q < \infty$ *is ensured by Step 1 above. Hence we deduce from the facts that*

$$T : L^2 \mapsto L^q(\mathbb{R}; L^r(\mathbb{R}^d)), \quad T^* : L^{\tilde{q}'}(\mathbb{R}; L^{\tilde{r}'}(\mathbb{R}^d)) \mapsto L^2(\mathbb{R}^d),$$

are linear bounded operators that $TT^* : L^{\tilde{q}'}(\mathbb{R}; L^{\tilde{r}'}(\mathbb{R}^d)) \mapsto L^q(\mathbb{R}; L^r(\mathbb{R}^d))$ *is linear bounded operator. By Christ-Kiselev's Lemma the truncated operator* $\widetilde{TT^*}g = \int_0^t S(t-t')g(t')dt'$ *also maps from* $L^{\tilde{q}'}(\mathbb{R}; L^{\tilde{r}'}(\mathbb{R}^d))$ *to* $L^q(\mathbb{R}; L^r(\mathbb{R}^d))$ *for any two admissible exponent pairs* (q,r), $(\tilde{q},\tilde{r})$ *with* $q, \tilde{q} \in (2,\infty)$.

[14.11.2018]

[16.11.2017]

2.2 L^2 theory

2.2.1 Local well-posedness in L^2

Theorem 2.2. *[LWP in* L^2*] Let* p *be an* L^2*-subcritical exponent, i.e.* $1 < p < 1 + \frac{4}{d}$. *Let* $\kappa = \pm 1$. *Let* $u_0 \in L^2(\mathbb{R}^d)$.

Then the Cauchy problem (NLS) *is locally well-posed LWP in* $L^2(\mathbb{R}^d)$ *in the following sense: There exist a positive time* $T > 0$ *depending on* $\|u_0\|_{L^2(\mathbb{R}^d)}, p, d$, *and a unique solution* $u = u(t,x)$ *defined on the time interval* $[-T,T]$ *such that*

$$u \in X_T := \left\{ u \in C([-T,T]; L^2(\mathbb{R}^d)) \,\middle|\, u \in L^q([-T,T]; L^{p+1}(\mathbb{R}^d)) \right\}$$

with admissible exponent pair $(q, p+1)$ *i.e.* $\dfrac{2}{q} + \dfrac{d}{p+1} = \dfrac{d}{2}$, $2 \leqslant q \leqslant \infty$,

and there exists a neighborhood U *of* u_0 *in* $L^2(\mathbb{R}^d)$ *such that*

$$\Phi : U \mapsto X_{T'}, \quad u_0 \mapsto u \text{ is Lipschitz continuous for any } T' < T.$$

Proof. We solve the integral equation (Duhamel) by searching for the fixed point of the mapping

$$\Psi : u \mapsto \Psi(u) = S(t)u_0 - i\kappa \int_0^t S(t-t')\big(|u(t')|^{p-1}u(t')\big)dt' \tag{2.39}$$

in the ball of the functional space X_T as

$$X_T(R) := \Big\{ u \in X_T \,\Big|\, \|u\|_{[-T,T]} := \|u\|_{L^\infty([-T,T];L^2(\mathbb{R}^d))} + \|u\|_{L^q([-T,T];L^{p+1}(\mathbb{R}^d))} \leqslant R \Big\}$$

with R, T to be determined later. We will use the Banach fixed-point theorem (contraction mapping theorem) in the complete metric space $(X_T(R), \|\cdot\|_{[-T,T]})$, and we shall prove that

- Ψ is a well-defined map in $X_T(R)$ with appropriately chosen R, T;
- Ψ is a contraction map in $X_T(R)$ for some small enough T.

Finally we conclude that there is a unique fixed point of Ψ and the flow map $\Psi : u_0 \mapsto u$ is Lipschitzian continuous from a neighborhood $U \subset L^2(\mathbb{R}^d)$ of u_0 to $X_T(R)$.

In the following C will denote some constant depending on p, d which may vary from line to line.

Step 1 Well-definedness of the map Ψ in $X_T(R)$

By Strichartz estimates in Theorem 2.1, we deduce that

$$\|S(t)u_0\|_{[-T,T]} \leqslant C(p,d)\|u_0\|_{L^2_x},$$

and

$$\begin{aligned}
\|\Psi(u) - S(t)u_0\|_{[-T,T]} &\leqslant C(p,d)\big\||u|^{p-1}u\big\|_{L^{q'}([-T,T];L^{(p+1)'}(\mathbb{R}^d))} \\
&\leqslant C\Big(\int_{-T}^T \big\||u|^{p-1}u\big\|_{L^{\frac{p+1}{p}}(\mathbb{R}^d)}^{q'} dt\Big)^{\frac{1}{q'}} \\
&= C\Big(\int_{-T}^T \|u\|_{L^{p+1}(\mathbb{R}^d)}^{pq'} dt\Big)^{\frac{1}{q'}}.
\end{aligned}$$

Since $1 < p < 1 + \frac{4}{d}$, we use Hölder's inequality $\|fg\|_{L^1} \leqslant \|f\|_{L^{\frac{q}{pq'}}} \|g\|_{L^{(1-\frac{pq'}{q})^{-1}}}$ to deduce

$$\begin{aligned}
\|\Psi(u)\|_{[-T,T]} &\leqslant C\|u_0\|_{L^2} + C\Big(\int_{-T}^T \|u\|_{L^{p+1}(\mathbb{R}^d)}^q dt\Big)^{\frac{p}{q}} T^\theta \\
&\leqslant C\|u_0\|_{L^2} + C\|u\|_{[-T,T]}^p T^\theta,
\end{aligned}$$

with $\theta = \frac{1}{q'}(1 - \frac{pq'}{q}) = \frac{1}{q'} - \frac{p}{q} = \frac{d}{4}(1 + \frac{4}{d} - p) > 0$.

We choose $R = 2C\|u_0\|_{L_x^2}$ and T_1 sufficiently small such that

$$C(2C\|u_0\|_{L^2})^p (T_1)^\theta = C\|u_0\|_{L^2}, \text{ i.e. } T_1 = (2C)^{-\frac{p}{\theta}} \|u_0\|_{L_x^2}^\beta$$
$$\text{with } \beta = \frac{1-p}{\theta} = \frac{4(1-p)}{d(1+\frac{4}{d}-p)} < 0,$$

and hence for any $T \leqslant T_1$,

$$\text{if } \|u\|_{[-T,T]} \leqslant R, \text{ then } \|\Psi(u)\|_{[-T,T]} \leqslant C\|u_0\|_{L^2} + C\|u_0\|_{L^2} = R,$$

and $S(t)u_0, \Psi(u) - S(t)u_0$ are continuous in $L^2(\mathbb{R}^d)$.

Step 2 Contraction map Ψ

Let $u, v \in X_T(R)$ and we calculate by Strichartz estimate

$$\begin{aligned}\|\Psi(u) - \Psi(v)\|_{[-T,T]} &= \Big\| \int_0^t S(t-t')\Big(|u(t')|^{p-1}u(t') - |v(t')|^{p-1}v(t')\Big) dt' \Big\|_{[-T,T]} \\ &\leqslant C \Big\| |u|^{p-1}u - |v|^{p-1}v \Big\|_{L^{q'}([-T,T];L^{(p+1)'}(\mathbb{R}^d))}.\end{aligned}$$

Since $||u|^{p-1}u - |v|^{p-1}v| \leqslant C_1(|u|^{p-1} + |v|^{p-1})|u-v|$ for some constant C_1, we proceed as in Step 1 to obtain

$$\begin{aligned}\|\Psi(u) - \Psi(v)\|_{[-T,T]} &\leqslant CC_1\Big(\int_{-T}^T \big(\|u\|_{L_x^{p+1}}^{(p-1)q'} + \|v\|_{L_x^{p+1}}^{(p-1)q'}\big)\|u-v\|_{L_x^{p+1}}^{q'} dt\Big)^{\frac{1}{q'}} \\ &\leqslant CC_1\big(\|u\|^{p-1}_{L^q([-T,T];L_x^{p+1})} + \|v\|^{p-1}_{L^q([-T,T];L_x^{p+1})}\big)\|u-v\|_{L^q([-T,T];L_x^{p+1})}T^\theta \\ &\leqslant C_2 R^{p-1}T^\theta \|u-v\|_{[-T,T]} \text{ for some constant } C_2 \geqslant C.\end{aligned}$$

Hence we take T such that

$$C_2 R^{p-1} T^\theta = \frac{1}{2} \text{ i.e. } T = (2C_2)^{-\frac{1}{\theta}}(2C)^{-\frac{p-1}{\theta}}\|u_0\|_{L_x^2}^\beta \leqslant T_1,$$

and the map Ψ is a contraction map on $X_T(R)$.

Step 3 Conclusion

By Banach fixed point theorem, there exists a unique fixed point $u \in X_T(R)$ of the map Ψ and hence $u \in X_T(R)$ solves uniquely (NLS) with $R = C_3\|u_0\|_{L_x^2}$, $T = C_3^{-1}\|u_0\|_{L_x^2}^\beta$ for some large enough constant C_3 (to be determined later). Without loss of generality we can assume that for any initial data in the neighborhood of u_0: $U = \{v_0 \in L^2(\mathbb{R}^d) | \|u_0 - v_0\|_{L^2(\mathbb{R}^d)} < \|u_0\|_{L^2(\mathbb{R}^d)}\}$ such that $\|v_0\|_{L^2(\mathbb{R}^d)} < 2\|u_0\|_{L^2(\mathbb{R}^d)}$, there is a unique solution $v \in X_T(R)$ of (NLS).

We are going to show the Lipschitz continuity of the flow map $\Phi : U \mapsto X_{T'}(R)$ via $u_0 \mapsto u$ for all $T' < T$. Let $u_0, v_0 \in U$ and we calculate

$$\begin{aligned}\Phi(u_0) - \Phi(v_0) &= S(t)(u_0 - v_0)\\ &\quad - i\kappa \int_0^t S(t-t')\Big(|\Phi(u_0)(t')|^{p-1}\Phi(u_0)(t') - |\Phi(v_0)(t')|^{p-1}\Phi(v_0)(t')\Big)dt'.\end{aligned}$$

As in Step 2, we derive that

$$\begin{aligned}\|\Phi(u_0) - \Phi(v_0)\|_{[0,T]} &\leqslant C\|u_0 - v_0\|_{L^2_x}\\ &\quad + C\big(\|\Phi(u_0)\|_{[0,T]}^{p-1} + \|\Phi(v_0)\|_{[0,T]}^{p-1}\big)\|\Phi(u_0) - \Phi(v_0)\|_{[0,T]}T^\theta\\ &\leqslant C\|u_0 - v_0\|_{L^2_x} + C_2\big(\|u_0\|_{L^2_x}^{p-1} + \|v_0\|_{L^2_x}^{p-1}\big)\|\Phi(u_0) - \Phi(v_0)\|_{[0,T]}T^\theta,\end{aligned}$$

such that

$$\|\Phi(u_0) - \Phi(v_0)\|_{[0,T]} \leqslant 2C_3\|u_0 - v_0\|_{L^2_x},$$

if $T = C_3^{-1}\|u_0\|_{L^2_x}^{\beta}$ for sufficiently large C_3. □

Remark 2.4. *The nonlinear Schrödinger equation* (NLS) *holds in the distribution sense: It follows from the Duhamel formulation* (Duhamel) *and the well-definedness of the nonlinearity when $u \in X_T$*

$$|u|^{p-1}u \in L^{\frac{q}{p}}([-T,T]; L^{\frac{p+1}{p}}(\mathbb{R}^d)), \quad \frac{q}{p} > q' \text{ if } p < 1 + \frac{4}{d}.$$

Then by Strichartz estimates the solution u itself belongs to any functional space $L^{\tilde q}([-T,T]; L^{\tilde r}(\mathbb{R}^d))$ for any admissible exponent pair $(\tilde q, \tilde r)$.

By Sobolev embedding $H^1(\mathbb{R}^d) \hookrightarrow L^{p+1}(\mathbb{R}^d) = \big(L^{\frac{p+1}{p}}(\mathbb{R}^d)\big)'$, $|u|^{p-1}u \in L^{\frac{q}{p}}([-T;T]; H^{-1}(\mathbb{R}^d))$ and hence the equation (NLS) *makes sense at least in $L^{q'}([-T,T]; H^{-2}(\mathbb{R}^d))$. However it is in generally not true that we can take the L^2_x-inner product between the equation* (NLS) *and the solution u directly to show the mass conservation law: That is why we first do regularization and then take the L^2_x inner product in next Subsection 2.2.2.*

2.2.2 Global well-posedness in L^2

Theorem 2.3. *[GWP in L^2] Let $1 < p < 1 + \frac{4}{d}$. The solution obtained in Theorem 2.2 exists globally in time such that*

$$u \in C(\mathbb{R}; L^2_x) \cap L^q_{\text{loc}}(\mathbb{R}; L^p_x) \text{ and } \|u(t,\cdot)\|_{L^2_x} = \|u_0\|_{L^2_x}, \forall t \in \mathbb{R}. \qquad (2.40)$$

Proof. We show the conservation of the L_x^2-norm, i.e. the mass conservation law (1.8), rigorously for $u \in X_T$ satisfying (NLS).

Step 1 Regularization

Take $\varphi \in C_0^\infty(\mathbb{R}^d)$ with $\varphi \geqslant 0$, $\int_{\mathbb{R}^d} \varphi(x)\,\mathrm{d}x = 1$. Denote $\varphi_n(x) = n^d\varphi(nx)$. Similarly we take $\psi \in C_0^\infty([-T,T])$, $\psi \geqslant 0$, $\int_{\mathbb{R}} \psi(t)dt = 1$ and denote $\psi_m(t) = m\psi(mt)$, $m \geqslant N$ with N sufficiently large. Since $u \in C([-T,T];L_x^2) \cap L^q([-T,T];L_x^{p+1})$, we have (**Exercise**.)

$$\begin{aligned}
&\psi_m * \varphi_n * u \to u \text{ in } C\big(I_N;L_x^2\big) \cap L^q\big(I_N;L_x^{p+1}\big),\\
&\psi_m * \varphi_n * (|u|^{p-1}u) \to (|u|^{p-1}u) \text{ in } L^{q'}(I_N;L_x^{(p+1)'})
\end{aligned}$$

as $m,n \to \infty$. Here we denote $I_N = (-(1-\frac{1}{N})T,(1-\frac{1}{N})T)$.

We take the convolution of (NLS) with φ_n and then with ψ_m to arrive at

$$i\partial_t u_{m,n} + \Delta u_{m,n} = \kappa\psi_m * \varphi_n * (|u|^{p-1}u), \quad u_{m,n} = \psi_m * \varphi_n * u. \tag{2.41}$$

We test the above equation for $u_{m,n}$ by $\overline{u_{m,n}} \in \mathcal{S}(I_N \times \mathbb{R}_x^d)$ and then take the imaginary part. Similarly as the derivation of (1.8), we derive

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d} |u_{m,n}|^2\,\mathrm{d}x = \kappa \mathrm{Im}\int_{\mathbb{R}^d}\big(\psi_m * \varphi_n * (|u|^{p-1}u)\big)\overline{u_{m,n}}\,\mathrm{d}x\,,$$

for all $t \in I_N$.

Step 2 Pass to the limit

For any $T' \in I_N$, we derive from the above equality that

$$\begin{aligned}
\frac{1}{2}(\|u_{m,n}(T')\|_{L_x^2} - \|u_{m,n}(0)\|_{L_x^2}) &= \kappa\mathrm{Im}\int_{[0,T']\times\mathbb{R}^d}\big(\psi_m * \varphi_n * (|u|^{p-1}u)\big)\overline{u_{m,n}}\,\mathrm{d}x\,dt\\
&\to \kappa\mathrm{Im}\int_{[0,T']\times\mathbb{R}^d}(|u|^{p-1}u)\overline{u}\,\mathrm{d}x\,dt = 0,
\end{aligned}$$

and hence

$$\|u(T')\|_{L_x^2} = \lim_{m,n\to\infty}\|u_{m,n}(T')\|_{L_x^2} = \lim_{m,n\to\infty}\|u_{m,n}(0)\|_{L_x^2} = \|u_0\|_{L_x^2}.$$

As the above holds for all $T' \in I_N$, it indeed holds for all $T' \in [-T,T]$.

Recall that the existence time T depends only on $p, d, \|u_0\|_{L_x^2}$, the solution obtained in Theorem 2.2 can be extended to all the time by uniqueness continuation. ☐

[16.11.2017]

[23.11.2018]

2.2.3 L^2 critical case

Let us consider the L^2 critical case $p = 1 + \frac{4}{d}$, which is quite interesting case: Recall the cubic nonlinear Schrödinger equations (1.30) and (1.35) with $(p, d) = (3, 2)$.

Theorem 2.4 (LWP & GWP for L^2 critical case)**.** *Let $p = 1 + \frac{4}{d}$ be the $L^2(\mathbb{R}^d)$ critical exponent. Let $\kappa = \pm 1$. Then*

- (NLS) *is locally well-posed in $L^2(\mathbb{R}^d)$ such that for any $u_0 \in L^2(\mathbb{R}^d)$, there exists a unique solution*

$$u \in X_T = \{u \in C([-T, T]; L^2(\mathbb{R}^d)) \mid u \in L^{p+1}_{t,x}([-T, T] \times \mathbb{R}^d)\},$$

  where $T > 0$ depending on u_0, d and there exists a neighborhood U of u_0 such that the flow map $\Phi : L^2 \mapsto X_T$ via $\Phi : u_0 \mapsto u$ is Lipschitz continuous;

- *There exists a sufficiently small constant $\varepsilon_0 > 0$ depending on d such that if $\|u_0\|_{L^2_x} \leqslant \varepsilon_0$ then* (NLS) *is globally well-posed in $L^2(\mathbb{R}^d)$ and the unique solution belongs to*

$$C(\mathbb{R}; L^2_x(\mathbb{R}^d)) \cap L^{p+1}_{t,x}(\mathbb{R} \times \mathbb{R}^d).$$

Sketchy proof. **Step 1 Smallness of $\|S(t)u_0\|_{L^{p+1}([-T_0,T_0]\times\mathbb{R}^d)}$ for small T_0** (**Exercise**.)For any $\varepsilon > 0$, for any $u_0 \in L^2$, there exists a neighborhood U of u_0 and $T_0 > 0$ such that

$$\|S(t)v_0\|_{L^{p+1}_{t,x}([-T_0,T_0]\times\mathbb{R}^d)} \leqslant \varepsilon, \quad \forall v_0 \in U.$$

Indeed, the mapping $S(t) : L^2(\mathbb{R}^d) \mapsto X_T$ is locally Lipschitz. Therefore we can take the neighborhood of u_0 as $U = \{v_0 \in L^2_x \mid \|v_0 - u_0\|_{L^2_x} < C^{-1}\varepsilon\}$ for sufficiently large C, and hence it remains to show the above for u_0.
Step 2 LWP in L^2
(**Exercise**.) We prove the local well-posedness result by searching for the fixed point for the mapping Ψ defined in (2.39) in

$$\{u \in X_{T_0} \mid \|u\|_{L^\infty_t([-T_0,T_0];L^2_x(\mathbb{R}^d))} \leqslant 2C\|u_0\|_{L^2_x}, \|u\|_{L^{p+1}_{t,x}([-T_0,T_0]\times\mathbb{R}^d)} \leqslant 2\varepsilon\},$$

for some sufficiently small ε depending on $\|u_0\|_{L^2_x}, d$ and T_0 depending on ε, u_0. By Step 1, we can assume that there exists T_0 such that for any initial data v_0 in the neighborhood U there exists a unique solution $v \in X_{T_0}$ with $\|v\|_{L^{p+1}_{t,x}([-T_0,T_0]\times\mathbb{R}^d)} \leqslant 2\varepsilon$.

(**Exercise**.) We prove that the flow map $\Phi : U \mapsto X_{T_0}$ is Lipschitz continuous.

Step 3 Small initial data case

(**Exercise**.) We prove the global well-posedness result in L^2 for small initial data $\|u_0\|_{L^2_x} \leqslant \varepsilon_0$, similarly as in Step 2. □

Remark 2.5.

- *We notice that in Theorem 2.4 there are well-posedness results in $L^2(\mathbb{R}^d)$ for the L^2-critical case if there are smallness conditions, either on the existing time or on the size of the initial data. Nevertheless here, since the existing time depends on u_0 itself and not only on its norm $\|u_0\|_{L^2_x}$, we can not use the mass conservation law to extend the local well-posed result to any time interval.*

- *In the defocusing L^2 critical case, there are some global well-posedness results without smallness assumption on the initial data, but*
  - under an additional decay assumption $|x|^m u_0 \in L^2(\mathbb{R}^d)$, $m > 3/5$, see [Bourgain 1998 JAM];
  - under an additional regularity assumption $u_0 \in H^s(\mathbb{R}^d)$, $s > 4/7$, see [Colliander-Keel-Staffilani-Takaoka-Tao 2008 DCDS-A];
  - in the radial case, see [Tao-Visan-Zhang 2007 DMJ], [Killip-Tao-Visan 2009 JEMS] for higher and two dimensional cases respectively.

  We are going to show the global well-posedness result in H^1_x for the defocusing H^1-subcritical case which includes the L^2-critical case.

- *In the focusing L^2 critical case, there is global well-posedness result if u_0 is radial and $\|u_0\|_{L^2} < \|Q\|_{L^2}$ where Q is the solution of the elliptic equation* (1.11)*. There may be blowup phenomena for $\|u_0\|_{L^2} \geqslant \|Q\|_{L^2}$.*

- *In the supercritical case $p > 1 + \frac{4}{d}$, there are ill-posedness results for* (NLS)*, see [Christ-Colliander-Tao 2003 arXiv]: If $s_c = \frac{d}{2} - \frac{2}{p-1} > 0$, then for any $s < s_c$, for any $0 < \delta, \epsilon < 1$ and any $t > 0$, there exist solutions u_1, u_2 of* (NLS) *with smooth initial data $u_1(0), u_2(0) \in \mathcal{S}$ such that*

$$\|u_1(0)\|_{H^s} + \|u_2(0)\|_{H^s} \leqslant C\epsilon, \quad \|u_1(0) - u_2(0)\|_{H^s} \leqslant C\delta,$$
$$\|u_1(t) - u_2(t)\|_{H^s} \geqslant c\epsilon.$$

  In the focusing case, the blowup phenominon in finite time from smooth data can be proved simply via the virial identity and we can construct the blowup example by applying scaling and Galilean transformation to the soliton solutions.

[23.11.2018]
[28.11.2018]

2.3 Sobolev spaces

2.3.1 Sobolev spaces $H^s(\mathbb{R}^d)$

Recall the definition (1.5) of the Sobolev space $H^s(\mathbb{R}^d)$, $s \in \mathbb{R}$ as follows

$$H^s(\mathbb{R}^d) := \{f \in \mathcal{S}'(\mathbb{R}^d) \mid \|f\|^2_{H^s(\mathbb{R}^d)} := \int_{\mathbb{R}^d} (1+|\xi|^2)^s |\hat{f}(\xi)|^2 \, \mathrm{d}\xi < \infty\}. \quad (2.42)$$

If $s \in \mathbb{N}$, then

$$H^s(\mathbb{R}^d) = \{f \in L^2(\mathbb{R}^d) \mid \partial^\alpha f \in L^2(\mathbb{R}^d), \forall \text{multi-index } \alpha \text{ with } |\alpha| \leqslant s\}.$$

It is easy to derive from the definition of $\|\cdot\|_{H^s}$-norm that $H^{s_1}(\mathbb{R}^d) \subset H^{s_0}(\mathbb{R}^d)$ if $s_0 \leqslant s_1$ and the following interpolation inequality by Hölder's inequality:

$$\|f\|_{H^{s_\theta}} \leqslant \|f\|^{1-\theta}_{H^{s_0}} \|f\|^\theta_{H^{s_1}}, \text{ with } s_\theta = (1-\theta)s_0 + \theta s_1. \quad (2.43)$$

The Sobolev space $H^s(\mathbb{R}^d)$ is a Hilbert space with the inner product

$$(u, v)_{H^s(\mathbb{R}^d)} = \int_{\mathbb{R}^d} (1+|\xi|^2)^s \hat{u}(\xi) \overline{\hat{v}(\xi)} \, \mathrm{d}\xi \, .$$

and it is isometrically anti-isomorphic to its dual space $(H^s(\mathbb{R}^d))'$. It is also very useful to identify $H^{-s}(\mathbb{R}^d)$ as the set of the continuous linear functionals on $H^s(\mathbb{R}^d)$ via $L^2(\mathbb{R}^d)$-inner product: Let $f \in \mathcal{S}'(\mathbb{R}^d)$ and $\hat{f} \in L^2_{\text{loc}}(\mathbb{R}^d)$, then

$$f \in H^{-s}(\mathbb{R}^d) \Leftrightarrow \sup{}_{g \in \mathcal{S}, \|g\|_{H^s} \leqslant 1} |\langle f, g \rangle_{\mathcal{S}', \mathcal{S}}| = \sup{}_{g \in \mathcal{S}, \|g\|_{H^s} \leqslant 1} \left| \int_{\mathbb{R}} f \bar{g} \, \mathrm{d}x \right| < \infty,$$

and we will denote $\langle f, g \rangle_{H^{-s}, H^s} = \langle (1+|\xi|^2)^{-s/2} \hat{f}, (1+|\xi|^2)^{s/2} \hat{g} \rangle_{L^2}$.

Theorem 2.5. *[Sobolev embedding for $H^s(\mathbb{R}^d)$] The following Sobolev embedding results hold true:*

- *If $0 \leqslant s < \frac{d}{2}$, then $H^s(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d)$ for any $p \in [2, p_c]$ with $\frac{d}{2} - s = \frac{d}{p_c}$ continuously and there exists a constant C depending on d, s such that*

$$\|f\|_{L^{p_c}(\mathbb{R}^d)} \leqslant C \|f\|_{\dot{H}^s(\mathbb{R}^d)}, \quad \forall f \in \mathcal{D}(\mathbb{R}^d), \quad (2.44)$$

where the homogeneous Sobolev norm is defined in (1.6)*: $\|f\|^2_{\dot{H}^s(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |\xi|^{2s} |\hat{f}(\xi)|^2 \, \mathrm{d}\xi$.*

- *If $s = \frac{d}{2}$, then $H^s(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d)$, for all $2 \leqslant p < \infty$ continuously;*
- *If $s > \frac{d}{2}$, then $H^s(\mathbb{R}^d) \hookrightarrow C_0(\mathbb{R}^d)$ continuously;*
- *If $0 \leqslant s < \frac{d}{2}$, then $L^{p'}(\mathbb{R}^d) \hookrightarrow H^{-s}(\mathbb{R}^d)$, $p \in [2, p_c]$ i.e. $p' \in [(\frac{1}{2}+\frac{s}{d})^{-1}, 2]$ continuously.*

Proof. **Step 1 Proof of** (2.44)

The case $s = 0$ is obvious and we consider the case $0 < s < \frac{d}{2}$, $p_c = \frac{2d}{d-2s} > 2$.

For any $A > 0$ we can decompose f into low- and high- frequency parts as follows:

$$f = f_l + f_h, \quad \hat{f}_l = \mathbf{1}_{<A}\hat{f}, \quad \hat{f}_h = \mathbf{1}_{\geqslant A}\hat{f}.$$

Then we can control the low frequency part f_l by

$$\begin{aligned}
\|f_l\|_{L^\infty} &\leqslant (2\pi)^{-\frac{d}{2}}\|\hat{f}_l\|_{L^1} = (2\pi)^{-\frac{d}{2}}\int_{|\xi|<A} |\hat{f}(\xi)||\xi|^s|\xi|^{-s}\,\mathrm{d}\xi \\
&\leqslant (2\pi)^{-\frac{d}{2}}\big\||\hat{f}(\xi)||\xi|^s\big\|_{L^2}\Big(\int_{|\xi|<A}|\xi|^{-2s}\,\mathrm{d}\xi\Big)^{\frac{1}{2}} \\
&\leqslant (2\pi)^{-\frac{d}{2}}\frac{\omega_d}{d-2s}\|f\|_{\dot{H}^s}A^{\frac{d}{2}-s} := C\|f\|_{\dot{H}^s}A^{\frac{d}{p_c}}.
\end{aligned}$$

We write

$$\|f\|_{L^p}^p = p\int_0^\infty \lambda^{p-1}|\{|f| \geqslant \lambda\}|d\lambda,$$

and we are going to estimate $|\{x \in \mathbb{R}^d \mid |f(x)| \geqslant \lambda\}|$ for each $\lambda \in (0,\infty)$. Indeed, for any $\lambda > 0$, we take $A = A(\lambda) = (4^{-1}C^{-1}\|f\|_{\dot{H}^s}^{-1}\lambda)^{\frac{p_c}{d}}$ such that the low frequency part $\|f_l\|_{L^\infty} \leqslant \lambda/4$ and hence

$$\begin{aligned}
|\{x \in \mathbb{R}^d \mid |f(x)| \geqslant \lambda\}| &\leqslant |\{x \in \mathbb{R}^d \mid |f_h(x)| \geqslant \lambda/2\}| \leqslant 4\lambda^{-2}\|f_h\|_{L^2}^2 = 4\lambda^{-2}\|\hat{f}_h\|_{L^2}^2 \\
&= 4\lambda^{-2}\int_{\left\{\xi\in\mathbb{R}^d \,\mid\, 4C\|f\|_{\dot{H}^s}|\xi|^{\frac{d}{p_c}} \geqslant \lambda\right\}} |\hat{f}|^2\,\mathrm{d}\xi\,.
\end{aligned}$$

Therefore

$$\begin{aligned}
\|f\|_{L^{p_c}}^{p_c} &= p_c\int_0^\infty \lambda^{p_c-1}|\{|f| \geqslant \lambda\}|d\lambda \\
&\leqslant 4p_c\iint_{\left\{\xi \,\mid\, 4C\|f\|_{\dot{H}^s}|\xi|^{\frac{d}{p_c}} \geqslant \lambda\right\}} \lambda^{p_c-3}|\hat{f}|^2\,\mathrm{d}\xi\,d\lambda \\
&\leqslant \frac{4p_c}{p_c-2}\int_{\mathbb{R}^d}(4C\|f\|_{\dot{H}^s}|\xi|^{\frac{d}{p_c}})^{(p_c-2)}|\hat{f}(\xi)|^2\,\mathrm{d}\xi \leqslant \frac{2d}{s}(4C)^{p_c-2}\|f\|_{\dot{H}^s}^{p_c}.
\end{aligned}$$

Step 2 Case $0 \leqslant s < \frac{d}{2}$

By interpolation of Lebesgue spaces and $\|f\|_{H^s} \sim \|f\|_{L^2} + \|f\|_{\dot{H}^s}$, we have the Sobolev embedding $H^s \hookrightarrow L^p$, $\forall p \in [2, p_c]$.

Step 3 Case $s = \frac{d}{2}$

For any $p \in [2, \infty)$, there exists $s_0 = \frac{d}{2} - \frac{d}{p} \in [0, \frac{d}{2})$ such that $H^{\frac{d}{2}} \hookrightarrow H^{s_0} \hookrightarrow L^p$.

Step 4 Case $s > \frac{d}{2}$

Since

$$\begin{aligned}\|\hat{f}\|_{L^1(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} (1 + |\xi|^2)^{s/2} |\hat{f}(\xi)| (1 + |\xi|^2)^{-s/2} \, \mathrm{d}\xi \\ &\leqslant \|f\|_{H^s} \big\|(1 + |\xi|^2)^{-s/2}\big\|_{L^2(\mathbb{R}^d)} \leqslant C\|f\|_{H^s}, \text{ if } s > \frac{d}{2},\end{aligned}$$

the function f as the inverse Fourier transform of a L^1-function is bounded, continuous and tends to 0 at infinity by Riemann-Lebesgue Lemma.

Step 5 Case $-s \in (-\frac{d}{2}, 0]$

By density, it suffices to show $\|f\|_{H^{-s}} \leqslant C\|f\|_{L^{p'}}$, $0 \leqslant \frac{d}{p'} - \frac{d}{2} \leqslant s$ for $f \in \mathcal{S}$. Indeed, since $\frac{d}{2} - \frac{d}{p} \leqslant s$, we derive from the embedding $H^s(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d)$ that

$$\|f\|_{H^{-s}} = \sup{}_{g \in \mathcal{S}, \|g\|_{H^s} \leqslant 1} \left| \int_{\mathbb{R}^d} f \bar{g} \, \mathrm{d}x \right| \leqslant \|f\|_{L^{p'}} \sup{}_{g \in \mathcal{S}, \|g\|_{H^s} \leqslant 1} \|g\|_{L^p} \leqslant C\|f\|_{L^{p'}}.$$

□

Remark 2.6. *Let* $s = 1$*, then*

$$\begin{aligned}&H^1(\mathbb{R}) \hookrightarrow C_0(\mathbb{R}), \quad H^1(\mathbb{R}^2) \hookrightarrow L^p(\mathbb{R}^2), \, \forall p \in [2, \infty), \\ &H^1(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3), \quad L^{p'}(\mathbb{R}^3) \hookrightarrow H^{-1}(\mathbb{R}^3), \, \forall p \in [2, 6].\end{aligned}$$

*(**Exercise**.) Find a function in* $H^1(\mathbb{R}^2)$ *but not in* $L^\infty(\mathbb{R}^2)$*.*

Corollary 2.1 (Gagliardo-Nirenberg's inequality)**.** *For any* $p \in [2, 2^*)$ *with* $2^* = \begin{cases} \infty & \text{if } d = 1, 2 \\ \frac{2d}{d-2} & \text{if } d \geqslant 3 \end{cases}$*, there exists a constant* C *depending on* p, d *such that the following interpolation inequality holds true*

$$\|f\|_{L^p(\mathbb{R}^d)} \leqslant C\|f\|_{L^2(\mathbb{R}^d)}^{1-\theta} \|\nabla f\|_{L^2(\mathbb{R}^d)}^{\theta}, \quad \forall f \in H^1(\mathbb{R}^d), \quad \theta = \frac{d}{2} - \frac{d}{p}.$$

Proof. It follows from the Sobolev embedding $\|f\|_{L^p(\mathbb{R}^d)} \leqslant C\|f\|_{\dot{H}^\theta(\mathbb{R}^d)}$ with $\theta = \frac{d}{2} - \frac{d}{p} \in [0, 1)$, $p \in [2, 2^*)$ and the Sobolev interpolation

$$\|f\|_{\dot{H}^\theta}^2 = \int_{\mathbb{R}^d} |\hat{f}|^{2(1-\theta)} (|\xi|^2 |\hat{f}|^2)^\theta \, \mathrm{d}\xi \leqslant \|f\|_{L^2}^{1-\theta} \|f\|_{\dot{H}^1}^\theta.$$

□

[28.11.2018]

[30.11.2018]

Theorem 2.6. *Let $s > 0$ and let $p_c = d(\frac{d}{2} - s)^{-1}$ if $s < \frac{d}{2}$ and $p_c = \infty$ if $s \geqslant \frac{d}{2}$. Then the embedding $H^s(\mathbb{R}^d) \hookrightarrow L^p_{\text{loc}}(\mathbb{R}^d)$, $1 \leqslant p < p_c$ is compact in the following sense: For any bounded sequence $(f_n)_n$ in $H^s(\mathbb{R}^d)$, there exists a subsequence $(f_{\psi(n)})_n$ and $f \in H^s(\mathbb{R}^d)$ such that for any compact set $K \subset\subset \mathbb{R}^d$*

$$f_{\psi(n)} \to f \text{ in } L^p(K).$$

Proof. (**Exercise**.)

Step 1 Take the smooth mollifier function: $\varphi \in C_0^\infty(\mathbb{R}^d)$, $\varphi \geqslant 0$, $\int_{\mathbb{R}^d} \varphi \, \mathrm{d}x = 1$ and its rescaled functions $\varphi_\varepsilon(x) = \varepsilon^{-d}\varphi(\varepsilon^{-1}x)$. Then

$$\sup_{\|g\|_{H^s} \leqslant 1} \|\varphi_\varepsilon * g - g\|_{L^2(\mathbb{R}^d)} \to 0 \text{ as } \varepsilon \to 0, \text{ if } s > 0,$$
$$\sup_{\|g\|_{H^s} \leqslant 1} \|\varphi_\varepsilon * g - g\|_{L^\infty(\mathbb{R}^d)} \to 0 \text{ as } \varepsilon \to 0, \text{ if } s > \frac{d}{2},$$

while for any $\varepsilon > 0$, $\sup_{\|g\|_{L^2} = 1} \|\varphi_\varepsilon * g - g\|_{L^2(\mathbb{R}^d)} \geqslant 1$.

Step 2 For any fixed $\varepsilon > 0$, for any fixed $R > 0$, the map

$$\varphi_\varepsilon * : L^2(\mathbb{R}^d) \mapsto L^\infty(\bar{B}_R), \quad \bar{B}_R = \{x \in \mathbb{R}^d \,|\, |x| \leqslant R\}$$

is compact (by Young's inequality and Arzela-Ascoli's theorem).

Step 3 The identity map

$$\text{Id} : H^s(\mathbb{R}^d)(\subset L^2(\mathbb{R}^d)) \mapsto L^2(\bar{B}_R)(\subset L^\infty(\bar{B}_R)), \quad s > 0$$

as the uniform limit of $\varphi_\varepsilon *$ is compact.

Step 4 Since $H^s(\mathbb{R}^d) \hookrightarrow L^{p_c}(\mathbb{R}^d)$ if $s < \frac{d}{2}$, then by interpolation (or Hölder's inequality) $H^s(\mathbb{R}^d) \hookrightarrow\hookrightarrow L^p(\bar{B}_R)$ compactly for all $p \in [1, p_c)$ and Cantor's diagonal argument ensures the compact embedding $H^s(\mathbb{R}^d) \hookrightarrow\hookrightarrow L^p_{\text{loc}}(\mathbb{R}^d)$. Similar result holds for $s \geqslant \frac{d}{2}$. □

Remark 2.7. *The compact embedding $H^s(\mathbb{R}^d) \hookrightarrow\hookrightarrow L^p_{\text{loc}}(\mathbb{R}^d)$ with $s \in [0, \frac{d}{2})$, $p \in [1, p_c)$ is optimal in the sense that the embeddings $H^s(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d)$, $2 \leqslant p < p_c$ and $H^s(\mathbb{R}^d) \hookrightarrow L^{p_c}_{\text{loc}}(\mathbb{R}^d)$ are not compact **(Exercise).** We are going to give the concentration-compactness lemma describing the embedding $H^1 \hookrightarrow L^2$ later in the lecture.*

2.3.2 Sobolev spaces $W^{k,p}(\mathbb{R}^d)$

We recall the Sobolev embedding results for the Sobolev spaces $W^{k,p}(\mathbb{R}^d)$ without proof. Recall the definition of the Sobolev space $W^{k,p}(\mathbb{R}^d)$, $k \geqslant 0$ integers as follows

$$W^{k,p}(\mathbb{R}^d) = \{f \in L^p(\mathbb{R}^d) | \partial^\alpha f \in L^p(\mathbb{R}^d), \, 0 \leqslant |\alpha| \leqslant k\}. \tag{2.45}$$

The Sobolev space $W^{k,p}(\mathbb{R}^d)$, $k \in \mathbb{N}$, $1 \leqslant p \leqslant \infty$ is a Banach space equipped with the norm

$$\|f\|_{W^{k,p}(\mathbb{R}^d)} = \begin{cases} \left(\sum_{|\alpha| \leqslant k} \|\partial^\alpha f\|^p_{L^p(\mathbb{R}^d)}\right)^{\frac{1}{p}} & \text{if } 1 \leqslant p < \infty, \\ \max_{|\alpha| \leqslant k} \|\partial^\alpha f\|_{L^\infty(\mathbb{R}^d)} & \text{if } p = \infty. \end{cases}$$

For $1 \leqslant p < \infty$, the test function space $\mathcal{D}(\mathbb{R}^d)$ is dense in $W^{k,p}(\mathbb{R}^d)$. If $p = 2$, then $W^{k,2}(\mathbb{R}^d) = H^k(\mathbb{R}^d)$ as defined in (2.42) and obviously $W^{k_1,p}(\mathbb{R}^d) \subset W^{k_0,p}(\mathbb{R}^d)$ if $k_0 \leqslant k_1$. We can also define the general Sobolev space $W^{s,p}(\Omega)$, $W^{s,p}_0(\Omega)$, $s \in \mathbb{R}$, $\Omega \subset \mathbb{R}^d$ some open set, which we don't discuss in this lecture. We just keep in mind that in bounded domains Ω, one has always to pay attention to the boundary.

Recall the definition of the Hölder spaces $C^{m,\sigma}(\Omega)$, $m \in \mathbb{N}$, $\sigma \in (0,1)$, $\Omega \subset \mathbb{R}^d$ some open set, as follows

$$\begin{aligned} &C^{m,\sigma}(\Omega) = \{f \in C^m(\Omega) \,|\, \partial^\alpha f \in C^\sigma(\Omega), \, \forall |\alpha| = m\}, \qquad (2.46) \\ &C^{0,\sigma}(\Omega) := C^\sigma(\Omega) = \{f \in C(\Omega) \,|\, \|f\|_{C^\sigma(K)} < \infty, \, \forall K \subset \Omega \text{ compact } \}, \end{aligned}$$

where

$$\|f\|_{C^\sigma(\overline{K})} = \|f\|_{L^\infty(K)} + \sup_{x,y \in K, \, x \neq y} \frac{|f(x) - f(y)|}{|x-y|^\sigma}.$$

Similarly we can define

$$\begin{aligned} &C^{m,\sigma}(\overline{\Omega}) = \{f \in C^m(\overline{\Omega}) \,|\, \partial^\alpha f \in C^\sigma(\overline{\Omega}), \, \forall |\alpha| = m\}, \\ &C^{0,\sigma}(\overline{\Omega}) := C^\sigma(\overline{\Omega}) = \{f \in C(\overline{\Omega}) \,|\, \|f\|_{C^\sigma(\bar{\Omega})} < \infty\}. \end{aligned}$$

We also have the following Sobolev embedding theorem for $W^{k,p}(\mathbb{R}^d)$ which we don't prove in this lecture:

Theorem 2.7. *Let $k \in \mathbb{N}^*$, $1 \leqslant p < \infty$. Then the following Sobolev embedding results hold true:*

- *If $1 \leqslant p < \frac{d}{k}$, then $W^{k,p}(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d)$ for any $q \in [p, p_c]$ with $\frac{d}{p} - k = \frac{d}{p_c}$ continuously and there exists a constant C depending on d, k, p such that*

$$\|f\|_{L^{p_c}(\mathbb{R}^d)} \leqslant C \sum_{|\alpha|=k} \|\partial^\alpha f\|_{L^p(\mathbb{R}^d)};$$

- *If* $p = \frac{d}{k}$, *then* $W^{k,\frac{d}{k}}(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d)$ *for any* $q \in [p, \infty)$ *continuously;*
- *If* $\max(1, \frac{d}{k}) < p < \infty$, *then* $W^{k,p}(\mathbb{R}^d) \hookrightarrow C^{m,\sigma}(\mathbb{R}^d) \cap C_0(\mathbb{R}^d)$, $m = [k - \frac{d}{p}]$, $\sigma = k - m$.

Furthermore, the embeddings are compact in the local sense as in Theorem 2.6: For example, let $k = 1$, $p^* = \begin{cases} \frac{dp}{d-p} & \text{if } p < d \\ \infty & \text{otherwise} \end{cases}$, *then the embedding* $W^{1,p}(\mathbb{R}^d) \hookrightarrow\hookrightarrow L^q_{\text{loc}}(\mathbb{R}^d)$ *is compact for* $q \in [1, p^*)$.

2.4 H^1 theory

2.4.1 Local well-posedness in H^1

Theorem 2.8 (LWP in H^1)**.** *Let* $1 < p < \infty$ *if* $d = 1, 2$ *and* $1 < p < 1 + \frac{4}{d-2}$ *if* $d \geqslant 3$ *be a* H^1 *subcritical exponent. Let* $\kappa = \pm 1$. *Let* $u_0 \in H^1(\mathbb{R}^d)$.

Then (NLS) *is locally well-posed in* $H^1(\mathbb{R}^d)$*: There exists a positive time* $T > 0$ *depending on* $\|u_0\|_{H^1}, p, d$, *a unique solution*

$$u \in Y_T = \{u \in C([-T, T]; H^1(\mathbb{R}^d)) \mid u \in L^q([-T, T]; W^{1,\rho}(\mathbb{R}^d))\}$$

with admissible exponent pair (q, ρ), *and there exists a neighborhood* V *of* u_0 *in* H^1 *such that the flow map*

$$\Phi : V \mapsto Y_T \text{ via } u_0 \mapsto u$$

is Lipschitzian continuous.

Proof. We are going to show that the nonlinear map $\Psi : u \mapsto \Psi(u)$ given by (2.39) is a well-defined contraction map in the complete metric space

$$Y_T(R) := \{u \in Y_T | \|u\|_T := \|u\|_{L^\infty_T H^1} + \|u\|_{L^q_T L^\rho} + \|\nabla u\|_{L^q_T L^\rho} \leqslant R\}$$

with appropriately chosen admissible exponent pair (q, ρ) and T, R (depending on $\|u_0\|_{H^1}, p, d$). Here we denote $\|u\|_{L^q_T Y} := \big\| \|u(t, \cdot)\|_Y \big\|_{L^q([-T,T])}$.

For $d \geqslant 3$, by Strichartz estimates,

$$\|\Psi(u)\|_{L^\infty_T L^2 \cap L^q_T L^\rho} \leqslant C\|u_0\|_{L^2} + C\||u|^{p-1}u\|_{L^{q'}_T L^{\rho'}}.$$

Similarly,

$$\begin{aligned}\|\nabla\Psi(u)\|_{L^\infty_T L^2 \cap L^q_T L^\rho} &\leqslant C\|\nabla u_0\|_{L^2} + C\|\nabla(|u|^{p-1}u)\|_{L^{q'}_T L^{\rho'}} \\ &\leqslant C\|\nabla u_0\|_{L^2} + C\Big\| \|u\|^{p-1}_{L^r} \|\nabla u\|_{L^\rho} \Big\|_{L^{q'}_T}, \quad \frac{p-1}{r} + \frac{1}{\rho} = \frac{1}{\rho'}.\end{aligned}$$

If $\frac{1}{r}=\frac{1}{\rho}-\frac{1}{d}\in(0,\frac{1}{\rho})$ such that the Sobolev embedding $\|u\|_{L^r(\mathbb{R}^d)}\leqslant C\|u\|_{W^{1,\rho}(\mathbb{R}^d)}$ holds, then we have

$$\|\nabla\Psi(u)\|_{L^\infty_TL^2\cap L^q_TL^\rho}\leqslant C\|\nabla u_0\|_{L^2}+CT^{\frac{1}{q'}-\frac{p}{q}}\|u\|_T^p,$$

if

$$\frac{2}{q}+\frac{d}{\rho}=\frac{d}{2},\quad \rho,q\geqslant 2,\quad \frac{p-1}{r}+\frac{1}{\rho}=\frac{1}{\rho'},\quad \frac{1}{r}=\frac{1}{\rho}-\frac{1}{d},$$
$$\text{such that } \frac{1}{q'}-\frac{p}{q}=\frac{d-2}{4}\big(1+\frac{4}{d-2}-p\big)>0 \text{ if } p<1+\frac{4}{d-2}.$$

Therefore for $(q,\rho)=\big(\frac{4(p+1)}{(d-2)(p-1)},\frac{d(p+1)}{d+p-1}\big)$ when $d\geqslant 3$, we arrive at

$$\|\Psi(u)\|_T=\|\Psi(u)\|_{L^\infty_TL^2\cap L^q_TL^\rho}+\|\nabla\Psi(u)\|_{L^\infty_TL^2\cap L^q_TL^\rho}\leqslant C\|u_0\|_{H^1}+CT^{\frac{1}{q'}-\frac{p}{q}}\|u\|_T^p,$$

and we can choose

$$R=C_1\|u_0\|_{H^1},\quad T=C_1^{-1}\|u_0\|_{H^1}^{-\frac{4}{d-2}\frac{p-1}{1+\frac{4}{d-2}-p}},$$

for some large enough constant C_1 such that Ψ is a contractive mapping in $Y_T(R)$. Since $|u|^{p-1}u\in L_t^{q'}([-T,T];W_x^{1,\rho'})$ with (q,ρ) the above admissible exponent pair, the unique fixed point indeed belongs to $L^{q_1}([-T,T];W^{1,\rho_1})$ for any admissible exponent pair (q_1,ρ_1) by Strichartz estimates.

For $d=1,2$, for any $1<p<\infty$, similarly as above we can show that the map Ψ is contractive in

$$\{u\in C([-T,T];H^1)\cap L^{q_0}([-T,T];W^{1,\rho_0})|\|u\|_T=\|u\|_{L^\infty_TH^1}+\|u\|_{L^{q_0}_TW^{1,\rho_0}}\leqslant R\}$$

for appropriately chosen admissible exponent pair (q_0,ρ_0) with $q_0>p\geqslant\rho_0/2>1$ and R,T. (**Exercise**.) □

[30.11.2018]

[07.12.2018]

2.4.2 Global well-posedness in H^1

Theorem 2.9 (GWP in H^1). *Assume the hypotheses in Theorem 2.8. Then the solution obtained in Theorem 2.8 can be extended uniquely globally in time if*

- *in the defocusing case $\kappa=1$;*

- *in the focusing case* $\kappa = -1$ *and* $1 < p < 1 + \frac{4}{d}$*;*
- *in the focusing case* $\kappa = -1$, $p = 1 + \frac{4}{d}$ *and* $\|u_0\|_{L^2} < c_0$ *with* c_0 *some fixed constant;*
- *in the focusing case* $\kappa = -1$, $1 + \frac{4}{d} < p$ *and* $\|u_0\|_{H^1} \leqslant \varepsilon_0$ *with* ε_0 *some sufficiently small constant,*

such that

$$u \in C(\mathbb{R}; H^1_x) \cap L^q_{\text{loc}}(\mathbb{R}; W^{1,\rho}(\mathbb{R}^d)) \text{ with admissible exponent pair } (q,\rho), \tag{2.47}$$

$$M(u(t)) = M(u_0), \quad E(u(t)) = E(u_0),\ \forall t \in \mathbb{R},$$

where

$$M(u) = \int_{\mathbb{R}^d} |u|^2 \,\mathrm{d}x\,, \quad E(u) = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla u|^2 \,\mathrm{d}x + \frac{\kappa}{p+1}\int_{\mathbb{R}^d} |u|^{p+1} \,\mathrm{d}x\,.$$

Proof. **Step 1** Mass and energy conservation laws on $[-T,T]$

Recall the proof of Theorem 2.3:

$$\operatorname{Im}\int_0^t \int_{\mathbb{R}^d} \bar{u}_{m,n} \cdot (\text{NLS})_{m,n} \Rightarrow M(u(t)) = M(u_0), \quad \forall t \in [-T,T],$$

where we made use of the following facts for some appropriate $r \in [2,\infty)$:

$$u \in C([-T,T]; L^2_x) \cap L^q([-T,T]; L^r_x), \quad |u|^{p-1}u \in L^{q'}([-T,T]; L^{r'}_x).$$

Recall in the proof of the H^1-LWP result in Theorem 2.8 that the solution satisfies

$$u \in C([-T,T]; H^1) \cap L^q_T W^{1,\rho}, \quad |u|^{p-1}u \in L^{q'}_T W^{1,\rho'},$$

which implies the mass conservation law immediately.

We follow the same procedure to show the conservation of the energy:

$$\operatorname{Im}\int_0^t \int_{\mathbb{R}^d} (-\Delta\bar{u}_{m,n} + \kappa|u_{m,n}|^{p-1}\bar{u}_{m,n}) \cdot (\text{NLS})_{m,n} \Rightarrow E(u(t)) = E(u_0),\ \forall t \in [-T,T].$$

Indeed, for $d \geqslant 3$, we have from the proof of Theorem 2.8, the Sobolev embedding results in Theorem 2.7 and the interpolation results in Lebesgue spaces (i.e. log-convexity of L^p norms) $\|f\|_{L^{p_\theta}} \leqslant \|f\|^{1-\theta}_{L^{p_0}}\|f\|^{\theta}_{L^{p_1}}$ if $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ that

$$u \in \left(L^\infty_T H^1 \cap L^q_T W^{1,\rho}\right) \subset \left(L^\infty_T L^{(\frac{1}{2}-\frac{1}{d})^{-1}} \cap L^q_T L^{(\frac{1}{\rho}-\frac{1}{d})^{-1}}\right) \subset L^{p\alpha}_T L^{p(\frac{1}{\rho}+\frac{1}{d})^{-1}},$$
$$|u|^{p-1}u \in L^{q'}_T W^{1,\rho'} \subset L^{q'}_T L^{(\frac{1}{\rho'}-\frac{1}{d})^{-1}},$$

for some

$$\alpha = \frac{q}{p}\frac{\frac{1}{2}-\frac{1}{\rho}}{(\frac{1}{2}-\frac{1}{d})-\frac{1}{p}(\frac{1}{\rho}+\frac{1}{d})} > q \text{ since } 1 < p < 1+\frac{4}{d-2}.$$

Then (**Exercise.**) we can assume

$u_{m,n} \to u$ in $L^q_T W^{1,\rho}$,

$|u_{m,n}|^{p-1}u_{m,n}$, $(|u|^{p-1}u)_{m,n} \to |u|^{p-1}u$ in $L^{q'}_T W^{1,\rho'} \cap L^{q'}_T L^{(\frac{1}{\rho'}-\frac{1}{d})^{-1}} \cap L^q_T L^{(\frac{1}{\rho}+\frac{1}{d})^{-1}}$,

which implies that if we take the limit in

$$\operatorname{Im} \int_0^t \int_{\mathbb{R}^d} (-\Delta \bar{u}_{m,n} + \kappa |u_{m,n}|^{p-1}\bar{u}_{m,n}) \cdot (\mathrm{NLS})_{m,n},$$

then $E(u(t)) = E(u_0)$, $\forall t \in [-T,T]$. Similarly we have the energy conservation law for $d = 1, 2$.

Step 2 If $\kappa = 1$, then recalling the mass and energy conservation laws we have the uniform bound

$$\|u(t)\|^2_{H^1_x} \leqslant M(u_0) + 2E(u_0)$$

on the time interval $[-T,T]$.

Since the existence time T only depends on $p, d, \|u_0\|_{H^1}$ and more precisely $T = C^{-1}\|u_0\|_{H^1}^{-\frac{4}{d-2}\frac{p-1}{1+\frac{4}{d-2}-p}}$ for some big enough constant C, the solution obtained in Theorem 2.8 can be extended uniquely to all the times.

Step 3 If $\kappa = -1$, then by the Gagliardo-Nirenberg's inequality in Corollary 2.1 for $p+1 < 2^*$, i.e. $p < 1 + \frac{4}{d-2}$ the H^1 subcritical exponent,

$$\|u\|_{L^{p+1}(\mathbb{R}^d)} \leqslant C_0 \|u\|^{1-\gamma}_{L^2(\mathbb{R}^d)} \|\nabla u\|^{\gamma}_{L^2(\mathbb{R}^d)}, \quad \gamma = \frac{d}{2} - \frac{d}{p+1} \in (0,1), \tag{2.48}$$

we obtain from the energy conservation law that

$$\begin{aligned}
\frac{1}{2}\|\nabla u(t)\|^2_{L^2_x(\mathbb{R}^d)} &\leqslant E(u_0) + \frac{1}{p+1}\|u(t)\|^{p+1}_{L^{p+1}_x(\mathbb{R}^d)} \\
&\leqslant E(u_0) + C\|u(t)\|^{(p+1)(1-\gamma)}_{L^2_x(\mathbb{R}^d)} \|\nabla u(t)\|^{(p+1)\gamma}_{L^2_x(\mathbb{R}^d)} \\
&\leqslant E(u_0) + C\|u(t)\|^{(p+1)(1-\gamma)(1-\frac{(p+1)\gamma}{2})^{-1}}_{L^2_x(\mathbb{R}^d)} + \frac{1}{4}\|\nabla u(t)\|^2_{L^2_x(\mathbb{R}^d)},
\end{aligned}$$

if

$$(p+1)\gamma = \frac{d}{2}(p+1) - d < 2 \text{ i.e. } 1 < p < 1 + \frac{4}{d}.$$

By the mass conservation law, we obtain the uniform bound on $\|u(t)\|_{H^1_x}$ on the existence time interval and hence the global well-posedness holds true in the mass subcritical case.

Step 4 If $\kappa = -1$ and $p = 1 + \frac{4}{d}$, then the above inequality is replaced by

$$\begin{aligned}\frac{1}{2}\|\nabla u(t)\|^2_{L^2_x(\mathbb{R}^d)} &\leqslant E(u_0) + \frac{1}{2+\frac{4}{d}}\|u(t)\|^{p+1}_{L^{p+1}_x(\mathbb{R}^d)}\\ &\leqslant E(u_0) + \frac{1}{2+\frac{4}{d}}C_0^{2+\frac{4}{d}}\|u(t)\|^{\frac{4}{d}}_{L^2_x(\mathbb{R}^d)}\|\nabla u(t)\|^2_{L^2_x(\mathbb{R}^d)}.\end{aligned}$$

Thus if $\|u_0\|_{L^2_x} < c_0$ some fixed constant (such that $\frac{1}{2+\frac{4}{d}}C_0^{2+\frac{4}{d}}c_0^{\frac{4}{d}} = \frac{1}{2}$) then the solution still extends globally in time.

Step 5 If $\kappa = -1$ and $p > 1 + \frac{4}{d}$ energy subcritical, then $(p+1)\gamma > 2$ and we can assume the smallness condition $\|u_0\|_{H^1_x} \leqslant \varepsilon_0$ such that $\|u(t)\|_{H^1_x} \leqslant 2\varepsilon_0$ globally in time for sufficiently small ε_0 (**Exercise**). □

Remark 2.8. *It was proved in [Weinstein '1983 CMP] that if $p = 1 + \frac{4}{d}$, then*

$$\begin{aligned}&\inf_{f\in H^1}\frac{\|f\|^{\frac{4}{d}}_{L^2(\mathbb{R}^d)}\|\nabla f\|^2_{L^2(\mathbb{R}^d)}}{\|f\|^{2+\frac{4}{d}}_{L^{2+\frac{4}{d}}(\mathbb{R}^d)}} = \frac{\|Q\|^{\frac{4}{d}}_{L^2(\mathbb{R}^d)}}{1+\frac{2}{d}},\\ &i.e.\ \frac{1}{2+\frac{4}{d}}\|f\|^{2+\frac{4}{d}}_{L^{2+\frac{4}{d}}(\mathbb{R}^d)} \leqslant \frac{1}{2}\frac{\|f\|^{\frac{4}{d}}_{L^2}}{\|Q\|^{\frac{4}{d}}_{L^2}}\|\nabla f\|^2_{L^2},\end{aligned} \tag{2.49}$$

where Q is the unique positive radial solution of (1.11). *It follows from* (2.49) *that*

$$E(u) = \frac{1}{2}\|\nabla u\|^2_{L^2} - \frac{1}{2+\frac{4}{d}}\|u\|^{2+\frac{4}{d}}_{L^{2+\frac{4}{d}}} \geqslant \frac{1}{2}\Big(1 - \frac{\|u\|^{\frac{4}{d}}_{L^2}}{\|Q\|^{\frac{4}{d}}_{L^2}}\Big)\|\nabla u\|^2_{L^2},$$

and in particular $E(Q) = 0$. If $\|u_0\|_{L^2(\mathbb{R}^d)} < \|Q\|_{L^2(\mathbb{R}^d)} = c_0$ then the focusing (NLS) *in the mass critical case is globally well-posed.*

Remark 2.9. *We can prove the local-in-time well-posedness results in $H^2(\mathbb{R}^d)$ for the case $\begin{cases}1 < p < \frac{d}{d-4} \text{ if } d \geqslant 5\\ 1 < p < \infty \text{ if } d \leqslant 4\end{cases}$, such that the solution stays in $C([-T,T]; H^2(\mathbb{R}^d)) \cap L^q([-T,T]; W^{2,p}(\mathbb{R}^d))$ with (q,p) admissible exponent pair.*

We can also consider the general Sobolev space $H^s(\mathbb{R}^d)$, $0 < s < \min\{1, \frac{d}{2}\}$ with $1 < p < 1 + \frac{4}{d-2s}$ and so on.

There are global well-posedness and scattering results for the energy-critical defocusing nonlinear Schrödinger equation: $p = 1 + \frac{4}{d-2}$, $d \geqslant 3$, $\kappa = 1$. See e.g. Colliander-Keel-Staffilani-Takaoka-Tao Ann. Math. 2008 for the case $d = 3$, $p = 5$.

Remark 2.10. *We can also show the existence result by* compactness method *(instead of Banach fixed point theorem here):*

- *Step 1: Construct a sequence of approximate smooth solutions u_ε (by regularising* (NLS)*);*
- *Step 2: Show a priori uniform estimates for the sequence u_ε (e.g. $\|u_\varepsilon\|_{L^p_T(X)} \leqslant C < \infty$);*
- *Step 3: Pass to the limit by some compactness argument which comes usually from the uniform bound for the time derivatives $\partial_t u_\varepsilon$, e.g. by Aubin-Lions' Lemma, if $X \hookrightarrow\hookrightarrow Y \hookrightarrow Z$, $\|u_\varepsilon\|_{L^p_T(X)} + \|\partial_t u_\varepsilon\|_{L^q_T(Z)} \leqslant C$, then $u_\varepsilon \to u$ in $L^p_T(Y)$ if $p < \infty$ or $u_\varepsilon \to u$ in $C([0,T];Y)$ if $p = \infty$ and $q > 1$, such that the strong limit u solves* (NLS).

The above procedure is a quite standard way to show the existence result, nevertheless the uniqueness/continuity results are not ensured a priori and their proofs need other arguments.

Here, we may follow the above procedure to show the well-posedness result for (NLS) *and we have used the idea to show the mass/energy conservation laws.*

The solutions obtained by contraction argument are usually called strong solutions *which are unique, continuously depending on the initial data, while the solutions obtained by the above compactness method are usually called* weak solutions *which could exist all the times but are possibly not unique. Sometimes the strong solutions and the weak solutions coincide.*

2.4.3 The virial space case

Let us define the virial space

$$\Sigma = \{u \in H^1(\mathbb{R}^d) \mid xu \in L^2(\mathbb{R}^d)\} = H^1(\mathbb{R}^d) \cap L^2(|x|^2 \,\mathrm{d}x), \tag{2.50}$$

consisting of H^1-functions which decay faster at infinity. We also define the associated norm as

$$\|u\|_\Sigma = (\|u\|^2_{H^1_x} + \|xu\|^2_{L^2_x})^{\frac{1}{2}}.$$

Define the partial differential operator P as

$$P = P(t) = x + 2it\nabla, \quad P_j = x_j + 2it\partial_{x_j}, \quad j = 1, \cdots, d.$$

It is easy to see that $P : \mathcal{S}(\mathbb{R}\times\mathbb{R}^d) \mapsto \mathcal{S}(\mathbb{R}\times\mathbb{R}^d)$ and by duality P is a map $\mathcal{S}'(\mathbb{R}\times\mathbb{R}^d) \mapsto \mathcal{S}'(\mathbb{R}\times\mathbb{R}^d)$. An easy calculation shows that

$$P(t)w = 2ite^{i\frac{|x|^2}{4t}}\nabla(e^{-i\frac{|x|^2}{4t}}w), \quad w \in \mathcal{S}'(\mathbb{R}\times\mathbb{R}^d), \quad t \neq 0. \tag{2.51}$$

Notice that P and $i\partial_t + \Delta$ commutes:

$$\begin{aligned}[P; i\partial_t + \Delta] &= (x + 2it\nabla)(i\partial_t + \Delta) - (i\partial_t + \Delta)(x + 2it\nabla)\\ &= -i\partial_t(2it)\nabla - 2\nabla = 0.\end{aligned}$$

If $u = S(t)g$, $g \in \Sigma$ solves the free Schrödinger equation $i\partial_t u + \Delta u = 0$, $u|_{t=0} = g$, then $P(t)u$ satisfies also the free Schrödinger equation with the initial data xg which itself has a unique solution $S(t)(xg)$. Therefore we have

$$P(t)S(t)g = (x + 2it\nabla)S(t)g = S(t)xg, \tag{2.52}$$

for $g \in \Sigma$. If $g \in \Sigma$, then $S(t)g \in \Sigma$ for any $t \in \mathbb{R}$:

$$xS(t)g = -2it\nabla S(t)g + S(t)(xg) \in L^2(\mathbb{R}^d), \quad \forall g \in \Sigma,$$

and $P(t)S(t)g = S(t)(xg) \in L^q_t(L^r)$ for (q, r) admissible exponent pair by Strichartz estimate.

We can also show $S(t) : \mathcal{S}(\mathbb{R}^d) \mapsto \mathcal{S}(\mathbb{R}^d)$, $t \in \mathbb{R}$ by use of (2.52). Indeed, $S(t) : H^\infty(\mathbb{R}^d) \mapsto H^\infty(\mathbb{R}^d)$, $H^\infty(\mathbb{R}^d) = \cap_{k\geqslant 0}H^k(\mathbb{R}^d)$ and hence by (2.52), for any $t \in \mathbb{R}$, for any $g \in \mathcal{S}(\mathbb{R}^d)$,

$$\begin{aligned}&xS(t)g = (P - 2it\nabla)S(t)g = S(t)(xg) - 2it\nabla S(t)g \in H^\infty(\mathbb{R}^d),\\ &x_jx_kS(t)g = x_j\big(S(t)(x_kg) - 2it\partial_{x_k}S(t)g\big)\\ &\qquad = S(t)(x_jx_kg) - 2it\partial_{x_j}S(t)(x_kg) - 2itx_j\partial_{x_k}S(t)g \in H^\infty(\mathbb{R}^d), \cdots\end{aligned}$$

Therefore (2.52) holds for all $g \in \mathcal{S}'(\mathbb{R}^d)$.

[07.12.2018]

[12.12.2018]

Theorem 2.10. *Let $p \in (1, 2^* - 1)$ be H^1 subcritial exponent. Let $\kappa = \pm 1$. Let $u_0 \in \Sigma$. Then the Cauchy problem* (NLS) *is locally well-posed in Σ, such that there exists a positive time T depending only on $\|u_0\|_{H^1}, p, d$, a unique solution $u \in C([-T, T]; \Sigma) \cap L^q([-T, T]; W^{1,\rho})$, $Pu \in L^q([-T, T]; L^\rho)$ with admissible exponent pair (q, ρ) and a neibourhood of u_0 in Σ such that the flow map is Lipschitz continuous on it. Furthermore, the global well-posedness result holds true under the four assumptions in Theorem 2.9.*

Sketchy proof. We apply $S(-t)$ on both the left and right sides of (2.52): $P(t)S(t) = S(t)x$ such that $S(-t)P(t) = xS(-t)$ and hence

$$\begin{aligned} P(t)S(t-t') &= P(t)S(t)S(-t') = S(t)xS(-t') \\ &= S(t)S(-t')P(t') = S(t-t')P(t') \text{ on } \mathcal{S}'(\mathbb{R}^d). \end{aligned}$$

Recalling the nonlinear mapping Ψ given in (2.39), if

$$u \in Z_T(R, R_1) = \{u \in C([-T,T];\Sigma) \mid \|u\|_{L^q_T W^{1,\rho}} \leqslant R, \ \|Pu\|_{L^q_T L^\rho} \leqslant R_1\},$$

for some admissible exponent pair (q, ρ) given in Theorem 2.8, then we derive that

$$\begin{aligned} P(t)\Psi u &= P(t)S(t)u_0 - i\kappa \int_0^t P(t)S(t-t')(|u|^{p-1}u)(t')dt' \\ &= S(t)(xu_0) - i\kappa \int_0^t S(t-t')P(t')(|u|^{p-1}u)(t')dt'. \end{aligned} \tag{2.53}$$

By virtue of (2.51), we derive for $t \neq 0$,

$$|P(|u|^{p-1}u)| = 2|t||\nabla(e^{-i\frac{|x|^2}{4t}}|u|^{p-1}u)| = 2|t|\left|\nabla\left(|e^{-i\frac{|x|^2}{4t}}u|^{p-1}(e^{-i\frac{|x|^2}{4t}}u)\right)\right|,$$

and hence

$$\|P(|u|^{p-1}u)\|_{L^{q'}_T L^{\rho'}} \leqslant \left\|2p|t||u|^{p-1}|\nabla(e^{-i\frac{|x|^2}{4t}}u)|\right\|_{L^{q'}_T L^{\rho'}},$$

which is, by virtue of (2.51) again, bounded by

$$\left\|p|u|^{p-1}|Pu|\right\|_{L^{q'}_T L^{\rho'}}.$$

As in the proof of Theorem 2.8, for $d \geqslant 3$ we have

$$\|P(|u|^{p-1}u)\|_{L^{q'}_T L^{\rho'}} \leqslant p\||u|^{p-1}Pu\|_{L^{q'}_T L^{\rho'}} \leqslant CT^{\frac{1}{q'}-\frac{p}{q}}\|u\|^{p-1}_{L^q_T(W^{1,\rho})}\|Pu\|_{L^q_T L^\rho},$$

and hence we can choose $R = C\|u_0\|_{H^1_x}$, $R_1 = C\|xu_0\|_{L^2_x}$, $T = C^{-1}\|u_0\|^{-\theta}_{H^1_x}$ for C sufficiently large such that Ψ is a contraction mapping in $Z_T(R, R_1)$. □

Remark 2.11. *Noticing* (2.51), *we can proceed by a recurrence argument to arrive at*

$$P_\alpha = (x + 2itD)^\alpha = (2it)^{|\alpha|}e^{i|x|^2/4t}D^\alpha(e^{-i|x|^2/4t}), \quad [P_\alpha; i\partial_t + \Delta] = 0.$$

Based on the property of the operator P_α, e.g. $d = 1$, $p = 3$,

$$\|P_m(|u|^2u)\|_{L^2_x} \leqslant C_m\|u\|^2_{L^\infty_x}\|P_m u\|_{L^2_x}, \quad \|u\|_{L^\infty_x} \leqslant t^{-\frac{1}{2}}\|Pu\|^{\frac{1}{2}}_{L^2_x}\|u\|^{\frac{1}{2}}_{L^2_x},$$

[Hayashi-Nakamitsu-Tsutsumi 1986-1988] proved that if p is an odd integer, $u_0 \in H^m(\mathbb{R}^d) \cap L^2(|x|^k \, \mathrm{d}x)$, $m \geqslant k$, then the regularity and the decay property are both preserved on the existence time interval. In particular, if $u_0 \in \mathcal{S}(\mathbb{R}^d)$, then the solution of (NLS) *$u(t,\cdot) \in \mathcal{S}(\mathbb{R}^d)$ on the existence time interval.*

3 Large time behaviour

3.1 Virial and Morawetz identities

We define the virial potential

$$V(u) = \int_{\mathbb{R}^d} |x|^2 |u|^2 \, \mathrm{d}x \,, \tag{3.1}$$

which averages the mass density (with the mass defined in (1.8)) against the weight function $|x|^2$. We define the associated Morawetz action

$$W(u) = \mathrm{Im} \sum_{j=1}^{d} \int_{\mathbb{R}^d} x_j (\bar{u} \partial_{x_j} u) \, \mathrm{d}x \equiv \mathrm{Im} \int_{\mathbb{R}^d} r(\bar{u}\partial_r u) \, \mathrm{d}x \,, \quad r = |x|, \tag{3.2}$$

which averages the momentum densities (with the momentum defined in (1.9)) against the weights (x_j).

Then we have the following Virial and Morawetz identities

Proposition 3.1. *Let $u(t,x)$ be a Schwartz solution of the Cauchy problem* (NLS)*. Then*

$$\frac{1}{4}\frac{d}{dt} V(u(t)) = W(u(t)), \tag{3.3}$$

and

$$\frac{1}{2}\frac{d}{dt} W(u(t)) = \int_{\mathbb{R}^d} |\nabla u|^2 \, \mathrm{d}x \, + \kappa \big(\frac{d}{2} - \frac{d}{p+1}\big) \int_{\mathbb{R}^d} |u|^{p+1} \, \mathrm{d}x \,. \tag{3.4}$$

Proof. **Exercise.** Making use of the Pohozaev's Identity

$$\begin{aligned} &\mathrm{Re} \int_{\mathbb{R}^d} \Delta \bar{u} \big(x \cdot \nabla u \big) \, \mathrm{d}x \, = (\frac{d}{2} - 1) \int_{\mathbb{R}^d} |\nabla u|^2 \, \mathrm{d}x \,, \quad \forall u \in \mathcal{S}(\mathbb{R}^d), \\ &\text{or equivalently, } \mathrm{Re} \int_{\mathbb{R}^d} \Delta \bar{u} \big(\frac{d}{2} u + x \cdot \nabla u \big) \, \mathrm{d}x \, = - \int_{\mathbb{R}^d} |\nabla u|^2 \, \mathrm{d}x \,, \end{aligned} \tag{3.5}$$

where $|\nabla u|^2 = \sum_{j=1}^d ((\partial_{x_j} \mathrm{Re}\, u)^2 + (\partial_{x_j} \mathrm{Im}\, u)^2)$. **(Exercise).**

□

Remark 3.1. *We can define instead the Virial potential and Morawetz action, with the weights $|x|^2, (x_j)$ in* (3.1) *and* (3.2) *replaced by the new weights $|x|, (\frac{x_j}{|x|})$ respectively:*

$$\mathcal{V}(u) = \int_{\mathbb{R}^d} |x| |u|^2 \, \mathrm{d}x \,,$$

$$\mathcal{W}(u) = \mathrm{Im} \sum_{j=1}^{d} \int_{\mathbb{R}^d} \frac{x_j}{|x|} (\bar{u} \partial_{x_j} u) \,\mathrm{d}x \,.$$

Then we have the following identities (Lin-Strauss' Morawetz Identities) for the Schwartz solution u of the Cauchy problem (NLS)*:*

$$\frac{1}{2}\frac{d}{dt}\mathcal{V}(u(t)) = \mathcal{W}(u(t)),$$
$$\frac{d}{dt}\mathcal{W}(u(t)) = \int_{\mathbb{R}^d} \frac{|\not\nabla u|^2}{|x|}\,\mathrm{d}x + \kappa \frac{2(d-1)(p-1)}{p+1} \int_{\mathbb{R}^d} \frac{|u|^{p+1}}{|x|}\,\mathrm{d}x - \frac{1}{4}\int_{\mathbb{R}^d} (\Delta^2|x|)|u|^2\,\mathrm{d}x \,,$$

where $\not\nabla := \nabla - \frac{x}{|x|}(\frac{x}{|x|}\cdot\nabla)$ *denotes the angular gradient.*

Corollary 3.1. *Let* $p \in (1, 2^* - 1)$ *be energy subcritical exponent. Let* $u_0 \in \Sigma$ *and let* $u \in C([-T,T];H^1)$, $T < \infty$ *be the solution of* (NLS)*. Then* $u \in C([-T,T];\Sigma) \cap L^q([-T,T];W^{1,\rho})$, $Pu \in L^q([-T,T];L^\rho)$ *for any admissible exponent pair* (q,ρ)*, and the mass and energy conservation laws as well as the virial and Morawetz identities* (3.3)-(3.4) *hold for u on the existence time interval* $[-T,T]$*: For any* $t \in [-T,T]$,

$$M(u(t)) = M(u_0), \quad E(u(t)) = E(u_0),$$
$$\frac{1}{4}V(u(t)) - \frac{1}{4}V(u_0) = \int_0^t W(u(t'))dt',$$
$$\frac{1}{2}W(u(t)) - \frac{1}{2}W(u_0) = \int_0^t \int_{\mathbb{R}^d} |\nabla u|^2\,\mathrm{d}x\,dt + \kappa(\frac{d}{2} - \frac{d}{p+1}) \int_0^t \int_{\mathbb{R}^d} |u|^{p+1}\,\mathrm{d}x\,dt.$$

Sketchy proof. Recalling the proof of Theorem 2.10, there exists $T_0 > 0$ depending only on $\|u\|_{L^\infty_T(H^1)}$ such that there exists a unique solution $\tilde{u} \in C([t_0-T_0, t_0+T_0];\Sigma) \cap L^q([t_0-T_0, t_0+T_0];W^{1,\rho})$, $P\tilde{u} \in L^q([t_0-T_0, t_0+T_0];L^\rho)$ for any $t_0 \in [-T,T]$, and hence by uniqueness $u = \tilde{u}$ on $[-T,T]$.

We do a regularisation argument and repeat the proof of Proposition 3.1 to arrive at the identities (3.3)-(3.4) for u on $[-T,T]$. (**Exercise**.) □

3.2 Blowup

Theorem 3.1 (Blowup for the focusing case)**.** *Let* $s_c \in [0,1)$*, i.e.* $1 + \frac{4}{d} \leqslant p < 2^* - 1$*. Let* $\kappa = -1$*. Let* $u_0 \in \Sigma$ *with the initial energy* $E(u_0) < 0$*.*

Then the unique solution $u(t,x)$ *obtained in Theorem 2.10 blows up in finite time, and more precisely there exists* $T^* < +\infty$ *such that*

$$\lim_{t \uparrow T^*} \|\nabla u(t)\|_{L^2(\mathbb{R}^d)} = +\infty.$$

Proof. We consider positive time in the following and the negative time can be treated similarly.

If the solution $u \in C((a,b); H^1(\mathbb{R}^d))$ on some time interval (a,b), then by the virial and Morawetz identities (3.3)-(3.4), we derive that

$$\begin{aligned}\frac{1}{16}\frac{d^2}{dt^2}V(u) &= \frac{1}{2}\int_{\mathbb{R}^d} |\nabla u|^2 \, \mathrm{d}x \, - \frac{1}{2}(\frac{d}{2} - \frac{d}{p+1})\int_{\mathbb{R}^d} |u|^{p+1} \, \mathrm{d}x \\ &= E(u) - \frac{1}{2}(\frac{d}{2} - \frac{d+2}{p+1})\int_{\mathbb{R}^d} |u|^{p+1} \, \mathrm{d}x \leqslant E(u_0) < 0,\end{aligned} \tag{3.6}$$

since $\frac{d}{2} - \frac{d+2}{p+1} = (d+2)(\frac{1}{2+\frac{4}{d}} - \frac{1}{p+1}) \geqslant 0$ if $p \geqslant 1 + \frac{4}{d}$ is mass supercritical. Hence if $u \in C([0,\infty); H^1(\mathbb{R}^d))$, then the time-dependent quantity $V(u(t)) = \int_{\mathbb{R}^d} |x|^2 |u|^2 \, \mathrm{d}x$ is below a parabola which is negative in finite positive time which is not possible. Thus u blows up at some finite positive time.

More precisely, if $p > 1+\frac{4}{d}$, we can calculate on the existence time interval

$$\begin{aligned}\frac{1}{16}\frac{d^2}{dt^2}V(u) &= \frac{1}{4}\frac{d}{dt}W(u(t)) \\ &= \Big[\frac{1}{2} - \frac{d}{4}(\frac{p+1}{2} - 1)\Big]\int_{\mathbb{R}^d} |\nabla u|^2 \, \mathrm{d}x \, + \frac{d}{2}(\frac{p+1}{2} - 1)E(u) \\ &< -\alpha \int_{\mathbb{R}^d} |\nabla u|^2 \, \mathrm{d}x \, ,\end{aligned}$$

where $\alpha = -\frac{1}{2} + \frac{d}{4}(\frac{p+1}{2} - 1) = \frac{d}{8}(p - 1 - \frac{4}{d}) > 0$.

If initially $W(u_0) < 0$, then $W(u(t)) < 0$. Thus $\frac{d}{dt}V(u) < 0$ and $V(u)(t) \leqslant V(u_0)$. Since

$$|W(u)(t)| = -W(u)(t) \leqslant \|ru\|_{L^2(\mathbb{R}^d)} \|\nabla u\|_{L^2(\mathbb{R}^d)} \leqslant (V(u_0))^{\frac{1}{2}} \|\nabla u\|_{L^2(\mathbb{R}^d)},$$

we derive that

$$\frac{1}{4}\frac{d}{dt}(-W(u)) > \alpha (V(u_0))^{-1}(-W(u))^2,$$

and hence

$$(V(u_0))^{\frac{1}{2}} \|\nabla u\|_{L^2(\mathbb{R}^d)} \geqslant -W(u) \geqslant \frac{V(u_0)(-W(u_0))}{V(u_0) + 4\alpha W(u_0) t}$$

from which we derive that $\lim_{t \uparrow T^*} \|\nabla u(t)\|_{L^2(\mathbb{R}^d)} = \infty$ with

$$T^* = (-4\alpha W(u_0))^{-1} V(u_0) = \frac{V(u_0)}{-4\alpha W(u_0)}.$$

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If initially $W(u_0) \geqslant 0$, then by virtue of (3.6):

$$\frac{1}{4}\frac{d}{dt}W(u) \leqslant E(u_0) < 0,$$

there exists a positive time t_0 such that $W(u)(t_0) < 0$ and we are in the previous case again. We can balance the times t_0 and T^* to show the blowup time to be order $\frac{V_0}{\sqrt{W_0^2 - \frac{4}{\alpha}V_0E_0} - W_0}$.

If $p = 1 + \frac{4}{d}$, then (3.6) gives

$$\frac{1}{16}\frac{d^2}{dt^2}V(u) = \frac{1}{4}\frac{d}{dt}W(u) = E(u_0) < 0.$$

Thus

$$W(u)(t) = W(u_0) + 4E(u_0)t, \quad V(u)(t) = V(u_0) + 4W(u_0)t + 8E(u_0)t^2,$$

and hence there exists a positive time $T^* > 0$ (of order $\frac{W_0 + \sqrt{W_0^2 - 2V_0E_0}}{-4E_0}$) such that $V(u)(T^*) = 0$. By the equality $\|f\|_{L^2(\mathbb{R}^d)}^2 = -\frac{1}{d}\sum_{j=1}^d \int_{\mathbb{R}^d} x_j \partial_{x_j}(|f|^2)\,\mathrm{d}x$ for $f \in \mathcal{S}(\mathbb{R}^d)$, we derive the Heisenberg's inequality

$$\|f\|_{L^2(\mathbb{R}^d)}^2 \leqslant \frac{2}{d}\sum_{j=1}^d \|x_j f\|_{L^2(\mathbb{R}^d)} \|\partial_{x_j} f\|_{L^2(\mathbb{R}^d)}, \quad \forall f \in \Sigma.$$

Therefore

$$0 < \|u_0\|_{L^2(\mathbb{R}^d)}^2 = \|u(t)\|_{L^2(\mathbb{R}^d)}^2 \leqslant \frac{2}{d}(V(u))^{\frac{1}{2}} \|\nabla u\|_{L^2(\mathbb{R}^d)},$$

which together with $V(u)(T^*) = 0$ implies $\lim_{t \to T^*} \|\nabla u\|_{L^2(\mathbb{R}^d)} = \infty$. □

Remark 3.2. *The time T^* gives indeed an upper bound for the life span and the solution may blow up before T^*. We can also make use of the norm $\|u\|_{L^q}$ for $q \geqslant p+1$ instead of $\|\nabla u\|_{L^2}$ in the estimate of the lifespan. Indeed it is cheap to see that $\frac{1}{p+1}\|u\|_{L^{p+1}}^{p+1} = \frac{1}{2}\|\nabla u\|_{L^2}^2 - E(u_0) \to \infty$ as $t \to T^*$.*

Remark 3.3. *In the mass critical case $p = 1 + \frac{4}{d}$, we can assume the following assumptions instead of $E(u_0) < 0$:*

- *$E(u_0) = 0$ and $W(u_0) < 0$;*

- $E(u_0) > 0$ *and* $W(u_0) < -\sqrt{E(u_0)V_0(u_0)}$.

Corollary 3.2. *Let* $d \geqslant 3$, $1+\frac{4}{d} \leqslant p < 1+\frac{4}{d-2}$ *and* $u_0 \in H^1(\mathbb{R}^d)$. *Let* $u(t,x)$ *be the solution of the Cauchy problem* (NLS) *satisfying* $\lim_{t\uparrow T^*} \|\nabla u(t)\|_{L^2(\mathbb{R}^d)} = +\infty$, *then*

$$\|\nabla u(t)\|_{L^2(\mathbb{R}^d)} \geqslant C_0(T^* - t)^{-(\frac{1}{p-1}-\frac{d-2}{4})}, \quad \forall t \in [0, T^*).$$

Proof. Recall the proof of Theorem 2.8. For any time $t_0 < T^*$ with $\|u(t_0)\|_{H^1(\mathbb{R}^d)} < \infty$, the solution u with $\|u\|_{L^\infty([t_0,t_0+T];H^1(\mathbb{R}^d))} \leqslant R := C\|u(t_0)\|_{H^1(\mathbb{R}^d)}$ exists at least on the time interval $[t_0, t_0+T]$, $T > 0$ with

$$T = C^{-1}\|u(t_0)\|_{H^1(\mathbb{R}^d)}^{-\frac{4}{d-2}\frac{p-1}{1+\frac{4}{d-2}-p}} = C^{-1}\|u(t_0)\|_{H^1(\mathbb{R}^d)}^{-\frac{1}{\frac{1}{p-1}-\frac{d-2}{4}}}.$$

Hence

$$T^* - t_0 > T \text{ i.e. } \|u(t_0)\|_{H^1(\mathbb{R}^d)} \geqslant C(T^* - t_0)^{-(\frac{1}{p-1}-\frac{d-2}{4})}.$$

□

Similar results hold for $d = 1, 2$ **Exercise**.

Remark 3.4 (Blow up rates for the case $p = 1 + \frac{4}{d}$, $\kappa = -1$).
Pseudo-conformal blow up rate

Recall the pseudoconformal invariance in the mass critical case that if $u = u(t,x)$ *is a solution of the nonlinear Schrödinger equation* (NLS), *then so is* $v(t,x) = \frac{e^{i|x|^2/4t}}{|t|^{\frac{d}{2}}} u(\frac{x}{t}, \frac{1}{t})$. *If* $u_0 \in \Sigma$, *then for any* $t \neq 0$, $v(t,\cdot) \in \Sigma$.

Let $u(t,x) = e^{it}Q(x)$ *be the solitary solution of the focusing* (NLS), *then* $v(t,x) = \frac{e^{i(|x|^2+4)/4t}}{|t|^{\frac{d}{2}}} Q(\frac{x}{t})$ *is also a solution in* Σ *for any* $t \neq 0$, *while blows up at* $t = 0$*:* $\|\nabla v(t)\|_{L^2(\mathbb{R}^d)} = O(1/t)$ *as* $t \to 0$.

Indeed [Merle 1993] showed that the above is the unique minimal mass blow up solution: Let $p = 1 + \frac{4}{d}$, $\kappa = -1$ *and* u *be the solution of* (NLS) *with the initial data* $u_0 \in H^1(\mathbb{R}^d)$ *and* $\|u_0\|_{L^2(\mathbb{R}^d)} = \|Q\|_{L^2(\mathbb{R}^d)}$. *If* $\lim_{t\uparrow T} \|\nabla u(t)\|_{L^2} = \infty$, *then up to the symmetries in Subsection 1.1.3*

$$u(t,x) = \frac{e^{i(|x|^2+4)/4(T-t)}}{(T-t)^{\frac{d}{2}}} Q(\frac{x}{T-t}).$$

Let $d = 1, 2$, $p = 1+\frac{4}{d}$, $\kappa = -1$, $w_0 \in H^1(\mathbb{R}^d)$ *such that* $\|w_0\|_{L^2} = \|Q\|_{L^2} + \varepsilon$ *and* $\lim_{t\uparrow T} \|\nabla w(t)\|_{L^2} = \infty$. *[Bourgain-Wang 1997] showed that* $w = u + \varphi$, *where* u *is as above and* φ *remains smooth after the blow up time.*

[Merle 1990 CMP] also proved that for any given $T > 0$, any set of fixed points $\{x_1, \cdots, x_k\}$ in $\mathbb{R}^d$, there exists an initial data u_0 such that the corresponding solution of the focusing mass critical (NLS) *blows up exactly at time T with the total mass concentrating at the points $\{x_1, \cdots, x_k\}$.*
log-log blow up rate

Corollary 3.2 implies that the blow up rate is at least $\|\nabla u(t)\|_{L^2(\mathbb{R}^d)} \geqslant C(T^* - t)^{-\frac{1}{2}}$ in the mass critical case. Indeed, the numerical simulation suggests the existence of solutions with log-log blow up rate $\big(\frac{\ln|\ln|T^*-t||}{T^*-t}\big)^{\frac{1}{2}}$. And when $d = 1$, [Perelman 2001] established the existence of a solution with log-log blow up rate.

[Raphaël 2005] proved that there is a universal gap between the above two blowup rates: Let $\|u_0\|_{L^2} \in (\|Q\|_{L^2}, \|Q\|_{L^2} + \varepsilon)$ for $\varepsilon \ll 1$. Let u be the corresponding blowup solution, then either u blows up at log-log rate, or u blows up faster than pseudo-conformal rate, i.e. $\|\nabla u(t)\|_{L^2(\mathbb{R}^d)} \geqslant C(T^* - t)^{-1}$. However, the existence of blowup solutions with blowup rate different from these two cases is still open.

3.3 Scattering

Theorem 3.2. *Let $1 + \frac{4}{d} \leqslant p < 2^* - 1$ and $\kappa = 1$. Let $u_0 \in \Sigma$ and $u \in C(\mathbb{R}; \Sigma)$ be the global-in-time solution of* (NLS) *given in Theorem 2.10. Then*

$$u \in C(\mathbb{R}; \Sigma) \cap L^q(\mathbb{R}; W^{1,\rho}(\mathbb{R}^d)), \text{ with } (q, \rho) \text{ admissible exponent pair,}$$

and u scatters at large time in the sense that there exist two functions $u_\pm \in \Sigma$ such that

$$\lim_{t \to \pm\infty} \|u(t, \cdot) - S(t)u_\pm\|_\Sigma = 0.$$

Proof. We just show the case $t \to +\infty$ and the case $t \to -\infty$ follows similarly.
Step 1 Pointwise decay

Consider the time-dependent function

$$\begin{aligned} F(t) &= \int_{\mathbb{R}^d} |xu + 2it\nabla u|^2 \,\mathrm{d}x + \frac{8t^2}{p+1} \int_{\mathbb{R}^d} |u|^{p+1} \,\mathrm{d}x \\ &= \int_{\mathbb{R}^d} |x|^2 |u|^2 \,\mathrm{d}x - 4t\mathrm{Im} \int_{\mathbb{R}^d} x \cdot \nabla u \bar{u} \,\mathrm{d}x + 8t^2 E(u) \\ &= V(u) - 4tW(u) + 8t^2 E(u_0), \end{aligned}$$

where $V(u), W(u), E(u)$ are the Virial potential, Morawetz action and the energy defined in (3.1), (3.2) and (1.10) respectively. By view of the virial and Morawetz identities (3.3)-(3.4), we have

$$\frac{d}{dt}F(t) = \frac{4d\,t}{p+1}[1+\frac{4}{d}-p]\int_{\mathbb{R}^d}|u|^{p+1}\,\mathrm{d}x \leqslant 0, \quad \text{if } p \geqslant 1+\frac{4}{d}.$$

Let $v(t,x) = e^{-i|x|^2/4t}u(t,x)$, then

$$Pu = (x+2it\nabla)u = 2ite^{i\frac{|x|^2}{4t}}\nabla v,$$

and we have

$$\begin{aligned} 8t^2E(v) &= 4t^2\int_{\mathbb{R}^d}|\nabla v|^2\,\mathrm{d}x + \frac{8t^2}{p+1}\int_{\mathbb{R}^d}|v|^{p+1}\,\mathrm{d}x = F(t) \\ &\leqslant F(0) = V(u_0). \end{aligned}$$

Hence we have the following pointwise in time decay rate

$$\|\nabla v(t)\|_{L^2(\mathbb{R}^d)} \leqslant (2t)^{-1}(V(u_0))^{\frac{1}{2}}.$$

By Gagliardo-Nirenberg's inequality in Corollary 2.1 and the mass conservation law

$$\|v(t)\|_{L^2_x} = \|u(t)\|_{L^2_x} = \|u_0\|_{L^2_x} = \|v_0\|_{L^2_x},$$

we have the following pointwise decay rate for $\|u\|_{L^r_x(\mathbb{R}^d)}$, $r \in [2, 2^*)$ (in comparison with (1.4) for the linear Schrödinger group $S(t)$)

$$\begin{aligned} \|u(t)\|_{L^r(\mathbb{R}^d)} = \|v(t)\|_{L^r(\mathbb{R}^d)} &\leqslant C\|v(t)\|_{L^2(\mathbb{R}^d)}^{1-(\frac{d}{2}-\frac{d}{r})}\|\nabla v(t)\|_{L^2(\mathbb{R}^d)}^{\frac{d}{2}-\frac{d}{r}} \\ &\leqslant C|t|^{-(\frac{d}{2}-\frac{d}{r})}\|u_0\|_{L^2(\mathbb{R}^d)}^{1-(\frac{d}{2}-\frac{d}{r})}(V(u_0))^{\frac{1}{2}(\frac{d}{2}-\frac{d}{r})}, \quad \forall r \in [2, 2^*). \end{aligned} \tag{3.7}$$

Step 2 Scattering in $L^2(\mathbb{R}^d)$

Recall the Duhamel's formula (Duhamel) for the globally defined solution $u(t,x)$ of (NLS). Then $w(t,\cdot) = S(-t)u(t,\cdot) \in H^1_x(\mathbb{R}^d)$ satisfies

$$w(t) = u_0 - i\int_0^t S(-t'')(|u|^{p-1}u)(t'')dt''.$$

Then for any $0 < t' < t$,

$$w(t) - w(t') = -i\int_{t'}^t S(-t'')(|u|^{p-1}u)(t'')dt'', \tag{3.8}$$

such that by Strichartz estimate (2.38)

$$\|w(t)-w(t')\|_{L^2(\mathbb{R}^d)} \leqslant C\||u|^{p-1}u\|_{L^{q'}([t',t];L^{r'})} = C\Big\|\|u\|^p_{L^{pr'}(\mathbb{R}^d)}\Big\|_{L^{q'}([t',t])}$$

where (q,r) could be any admissible exponent pair. By use of the pointwise decay (3.7) in Step 1, we choose $r = p+1 < 2^*$, $\frac{2}{q} = \frac{d}{2} - \frac{d}{p+1}$ to arrive at

$$\|w(t) - w(t')\|_{L^2(\mathbb{R}^d)} \leqslant C_0 \Big\|(t'')^{-(\frac{d}{2}-\frac{d}{p+1})p}\Big\|_{L^{q'}([t',t])} = C_0 \Big(\int_{t'}^{t} (t'')^{-\frac{2pq'}{q}} dt''\Big)^{\frac{1}{q'}},$$

where C_0 is some constant depending on the initial data $\|u_0\|_\Sigma$. If $p \geqslant 1 + \frac{4}{d}$ then $q \leqslant 2 + \frac{4}{d}$ such that $\frac{2p}{q} > 1$. Hence $\|w(t) - w(t')\|_{L^2(\mathbb{R}^d)} \to 0$ whenever $t', t \to \infty$. Therefore there exists $u_+ \in L^2(\mathbb{R}^d)$ such that $\|u(t) - S(t)u_+\|_{L^2(\mathbb{R}^d)} = \|w(t) - u_+\|_{L^2(\mathbb{R}^d)} \to 0$ as $t \to +\infty$.

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Step 3 Scattering in $H^1(\mathbb{R}^d)$

We first claim that $u \in L^q([0,\infty); W^{1,p+1})$. Indeed, we have already shown $u \in L^q_{\text{loc}}(\mathbb{R}; W^{1,p+1})$ in Theorem 2.9 such that $\|u\|_{L^q([0,T];W^{1,p+1})} \leqslant C(T) < \infty$ for any finite time $T > 0$. For any $t \geqslant T > 0$, by applying Strichartz estimates on the Duhamel's formula (Duhamel) (and also on the spatial derivative of (Duhamel)), we have

$$\begin{aligned}
\|u\|_{L^q([0,t];W^{1,p+1})} &\leqslant C\|u_0\|_{H^1_x} + C\big\||u|^{p-1}u\big\|_{L^{q'}([0,T];W^{1,\frac{p+1}{p}})} + C\big\||u|^{p-1}u\big\|_{L^{q'}([T;t];W^{1,\frac{p+1}{p}})} \\
&\leqslant C\|u_0\|_{H^1_x} + CT^{\frac{1}{q'}-\frac{p}{q}}\|u\|^p_{L^q([0,T];W^{1,p+1})} + C\big\|\|u\|^{p-1}_{L^{p+1}_x}\|u\|_{W^{1,p+1}_x}\big\|_{L^{q'}([T,t])} \\
&\leqslant C\|u_0\|_{H^1_x} + CT^{1-\frac{p+1}{q}}\|u\|^p_{L^q([0,T];W^{1,p+1})} \\
&\quad + C_0\Big\|(t'')^{-\frac{2}{q}(p-1)}\Big\|_{L^{(\frac{1}{q'}-\frac{1}{q})^{-1}}([T,t])}\|u\|_{L^q([T,t];W^{1,p+1}_x)},
\end{aligned}$$

where we used the pointwise decay estimate (3.7) for the last inequality and we now calculate

$$\Big\|(t'')^{-\frac{2}{q}(p-1)}\Big\|_{L^{(\frac{1}{q'}-\frac{1}{q})^{-1}}([T,t])} = C(T^{-\theta} - t^{-\theta}) \leqslant CT^{-\theta}$$
$$\text{with } \theta = \frac{2}{q}(p-1) - (\frac{1}{q'} - \frac{1}{q}) = \frac{2}{q}p - 1 > 0 \text{ when } p \geqslant 1 + \frac{4}{d}.$$

Hence by choosing T large enough such that $C_0CT^{-\theta} \leqslant \frac{1}{2}$, we have

$$\|u\|_{L^q([0,t];W^{1,p+1})} \leqslant C\|u_0\|_{H^1_x} + CT^{1-\frac{p+1}{q}}\|u\|^p_{L^q([0,T];W^{1,p+1})} \leqslant C(T) < \infty,$$

and as $t \to \infty$ we derive $u \in L^q([0,\infty); W^{1,p+1})$.

We apply spatial derivative and then Strichartz estimate and finally the decay rate (3.7) to (3.8), to arrive at

$$\begin{aligned}\|\nabla(w(t)-w(t'))\|_{L^2(\mathbb{R}^d)} &\leqslant C\Big\|\|u\|^{p-1}_{L^{p+1}(\mathbb{R}^d)}\|\nabla u\|_{L^{p+1}(\mathbb{R}^d)}\Big\|_{L^{q'}([t',t])}\\ &\leqslant C_0\Big\|(t'')^{-\frac{2}{q}(p-1)}\Big\|_{L^{(\frac{1}{q'}-\frac{1}{q})^{-1}}([t',t])}\|\nabla u\|_{L^q([t',t];L^{p+1}(\mathbb{R}^d))}\end{aligned}$$

which tends to zero whenever $t',t\to\infty$. Therefore $u_+\in H^1(\mathbb{R}^d)$ such that $\|u(t)-S(t)u_+\|_{H^1(\mathbb{R}^d)}=\|w(t)-u_+\|_{H^1(\mathbb{R}^d)}\to 0$ as $t\to+\infty$.
Step 4 Scattering in Σ

Recall (2.53) when we apply the operator $P=x+2it\nabla$ to (Duhamel). (**Exercise**.) Then the same argument as in Step 3 implies that

$$\|(x+2it\nabla)u\|_{L^q([0,\infty);L^{p+1})}<+\infty,$$

and hence by $xS(-t)=S(-t)P(t)$, we arrive at from (3.8) that

$$\begin{aligned}\|(xw)(t)-(xw)(t')\|_{L^2(\mathbb{R}^d)} &= \Big\|\int_{t'}^{t} S(-t')\big(P(|u|^{p-1}u)\big)(t'')dt''\Big\|_{L^2(\mathbb{R}^d)}\\ &\leqslant C\big\||u|^{p-1}|Pu|\big\|_{L^{q'}([t',t];L^{(p+1)'})}\to 0 \text{ as } t',t\to\infty.\end{aligned}$$

Therefore $\|xw(t)-xu_+\|_{L^2_x(\mathbb{R}^d)}\to 0$ as $t\to\infty$. □

Remark 3.5. *For $p\in(1,1+\frac{4}{d})$, then we also have the pointwise decay*

$$\|u(t)\|_{L^r(\mathbb{R}^d)}\leqslant C|t|^{-(\frac{d}{2}-\frac{d}{r})(1-\alpha(r))},\quad \alpha(r)=\begin{cases}0 \text{ if } 2\leqslant r\leqslant p+1,\\ \frac{(r-p-1)(4-d(p-1))}{(r-2)(4-(d-2)(p-1))} \text{ if } r>p+1,\end{cases}$$

by considering the time-dependent quantity $t^2\int_{\mathbb{R}^d}|v|^{p+1}\,\mathrm{d}x$ and then the quantity $t^2\int_{\mathbb{R}^d}|\nabla v|^2\,\mathrm{d}x$ via the equality

$$\frac{d}{dt}(8t^2E(v))=\frac{d}{dt}F(u)=\frac{4dt}{p+1}(1+\frac{4}{d}-p)\int_{\mathbb{R}^d}|v|^{p+1}\,\mathrm{d}x\,.$$

And if we assume furthermore $p>(2+d+\sqrt{d^2+12d+4})/(2d)$ such that $2p>q$, then the above scattering result also holds true.

Let us give some further remarks concerning the exponent regime of p in the scattering theory:

- *We can relax the restriction on $p\in(1,2^*-1)$ for the scattering results, nevertheless there are no scattering theory in L^2_x if $p\leqslant 1+\frac{2}{d}$;*

- *For $p \in (1+\frac{2}{d}, 2^*-1)$, $\kappa = 1$, there exist scattering states in L^2_x, nevertheless if $p \leqslant \frac{2+d+\sqrt{d^2+12d+4}}{2d}$ and u_0 is large or if $p \leqslant 1+\frac{4}{d+2}$ we do not know whether $u_\pm \in \Sigma$;*
- *There is also scattering theory for the focusing case if $p \in (1+\frac{4}{d+2}, 1+\frac{4}{d})$, nevertheless if $p < 1+\frac{4}{d+2}$ there is no scattering theory in L^2;*
- *We can relax the restriction on the initial data, e.g. $u_0 \in H^1(\mathbb{R}^d)$ such that the scattering theory in the energy space $H^1(\mathbb{R}^d)$ holds for $p \in (1+\frac{4}{d}, 2^*-1)$, $d \geqslant 3$, $\kappa = 1$.*

We introduce briefly here the basic notions of scattering theory. Let X be a Banach space. Let $\mathcal{R}_\pm$ be the following two subsets in X:

$$\begin{aligned}\mathcal{R}_\pm = \{\varphi \in X \,|\, &\text{(NLS) with initial data } \varphi \text{ has a unique solution} \\ &u \text{ defined for all } t \geqslant 0 (t \leqslant 0) \text{ such that } u_\pm = \lim_{t\to\pm\infty} S(-t)u(t) \text{ exists in } X\},\end{aligned}$$

and we call $u_\pm$ the scattering states of φ at $\pm\infty$. Let $U_\pm$ be the following two operators

$$U_\pm : \mathcal{R}_\pm \mapsto X \text{ via } U_\pm(\varphi) = u_\pm = \lim_{t\to\pm\infty} S(-t)u(t).$$

If the mapping $U_\pm$ are injective, we define the wave operators

$$\Omega_\pm = (U_\pm)^{-1} : \mathcal{U}_\pm \mapsto \mathcal{R}_\pm, \quad \mathcal{U}_\pm = U_\pm(\mathcal{R}_\pm) \text{ via } \Omega_\pm(u_\pm) = \varphi.$$

Let $\mathcal{O}_\pm = U_\pm(\mathcal{R}_+ \cap \mathcal{R}_-)$ and we define the scattering operator $\mathbb{S}$

$$\mathbb{S} = U_+\Omega_- : \mathcal{O}_- \mapsto \mathcal{O}_+ \text{ via } \mathbb{S}u_- = u_+.$$

Notice that

$$\mathcal{R}_- = \overline{\mathcal{R}_+} := \{\varphi \,|\, \bar{\varphi} \in \mathcal{R}_+\}, \quad \mathcal{U}_- = \overline{\mathcal{U}_+}, \quad \mathcal{O}_- = \overline{\mathcal{O}_+},$$

and if $\kappa = 0$ the linear Schrödinger equation, then $U_\pm = \Omega_\pm = \mathbb{S} = \mathrm{Id}$.

Let $X = \Sigma$ and

$$1 + \frac{4}{d} \leqslant p < 2^* - 1, \quad \kappa = +1. \tag{3.9}$$

Then by Theorem 3.2,

$$\mathcal{R}_\pm = \Sigma, \quad U_\pm : \Sigma \mapsto \Sigma, \quad u_\pm = U_\pm(u_0) = u_0 - i\int_0^{\pm\infty} S(-t')(|u|^{p-1}u)(t')dt',$$

where $u(t)$ is the solution of (NLS) with initial data $u_0 \in \Sigma$. Inversely we have the wave operators $\Omega_\pm : u_\pm \to u_0$ as follows

Theorem 3.3. *Assume* (3.9)*, then for any* $u_+ \in \Sigma$ *(resp.* $u_- \in \Sigma$*), there exists a unique* $u_0 \in \Sigma$ *such that the Cauchy problem* (NLS) *with the initial data* u_0 *has a unique solution* $u \in C(\mathbb{R}; \Sigma)$ *with* $\|S(-t)u(t) - u_+\|_\Sigma \to 0$ *(resp.* $\|S(-t)u(t) - u_-\|_\Sigma \to 0$*) as* $t \to \infty$ *(resp.* $t \to -\infty$*).*

Hence we can define the scattering operator $\mathbb{S} : \Sigma \mapsto \Sigma$, $\mathbb{S}u_- = U_+\Omega_-u_-$.

Proof. Let $u_+ \in \Sigma$ and we follow the proof of Theorems 2.8 and 2.10, to search for the fixed point of the nonlinear map [3]

$$\Psi_+ : u \mapsto S(t)u_+ + i \int_t^\infty S(t-t')(|u|^{p-1}u)(t')dt',$$

in the following complete metric space with (q, ρ) admissible exponent pair:

$$\tilde{X}_T = \{u \in C([T,\infty); H^1(\mathbb{R}^d)) \mid Pu \in C([T,\infty); L^2(\mathbb{R}^d)),$$
$$\|u\|_{\tilde{X}_T} := \|u\|_{L^q([T,\infty);W^{1,\rho}(\mathbb{R}^d))} + \|Pu\|_{L^q([T,\infty);L^\rho(\mathbb{R}^d))} + \sup{}_{t\geqslant T}|t|^{\frac{2}{q}}\|u(t)\|_{L^\rho(\mathbb{R}^d)} \leqslant R\},$$

for some appropriately chosen R, T. Indeed, if $u_+ \in \Sigma$, then by Strichartz estimates and $P(t)S(t) = S(t)x$, the solution $w_+ = S(t)u_+ \in C(\mathbb{R}; \Sigma)$ satisfies

$$\|w_+\|_{L^q(\mathbb{R};W^{1,\rho}(\mathbb{R}^d))} + \|Pw_+\|_{L^q(\mathbb{R};L^\rho(\mathbb{R}^d))} \leqslant C\|u_+\|_\Sigma.$$

Since $Pw_+ = 2ite^{i|x|^2/4t}\nabla(e^{-i|x|^2/4t}w_+)$, we derive

$$\|\nabla(e^{-i|x|^2/4t}w_+)\|_{L^2(\mathbb{R}^d)} \leqslant (2|t|)^{-1}\|Pw_+\|_{L^2_x} = (2|t|)^{-1}\|xu_+\|_{L^2_x},$$

which, together with $\|e^{-i|x|^2/4t}w_+\|_{L^2(\mathbb{R}^d)} = \|w_+\|_{L^2(\mathbb{R}^d)} = \|u_+\|_{L^2_x}$ and Gagliardo-Nirenberg's inequality, implies

$$\|w_+(t)\|_{L^\rho_x} = \|e^{-i|x|^2/4t}w_+\|_{L^\rho_x} \leqslant C|t|^{-(\frac{d}{2}-\frac{d}{\rho})}(\|u_+\|_{L^2_x} + \|xu_+\|_{L^2_x}) \leqslant C|t|^{-\frac{2}{q}}\|u_+\|_\Sigma.$$

Hence $S(t)u_+ \in \tilde{X}_T$ if $R \geqslant C\|u_+\|_\Sigma$ for some constant C. Similarly we can consider the nonlinear term in the map Ψ_+, by use of Strichartz estimate and Hölder's inequality. (**Exercise**.) Therefore we can choose $R = C\|u_+\|_\Sigma$ and $T = C\|u_+\|_\Sigma^{\frac{p-1}{\frac{2p}{q}-1}}$ with C large enough such that Ψ_+ is a contraction mapping in $\tilde{X}_T$ and hence there exists a unique fixed point u of Ψ_+ in $\tilde{X}_T$.

[3] The formulation of the nonlinear map Ψ_+ is motivated by taking the difference between the Duhamel's formula for $u(t) = S(t)u_0 - i\int_0^t S(t-t')(|u|^{p-1}u)(t')dt'$ and $S(t)u_+ = S(t)u_0 - i\int_0^\infty S(t-t')(|u|^{p-1}u)(t')dt'$.

For the fixed point $u \in C([T,\infty);\Sigma)$, $u(T) \in \Sigma$, by the formulation of Ψ_+, we have [4]

$$u(t+T) = S(t)u(T) - i\int_0^t S(t-t')(|u|^{p-1}u)(T+t')dt',$$

that is, $u_T(t,x) := u(T+t,x)$ is the solution of defocusing nonlinear Schrödinger equation (NLS) with the initial data $u_T(0) = u(T) \in \Sigma$ which hence exists globally in time $u_T \in C(\mathbb{R};\Sigma) \cap L^q(\mathbb{R};W^{1,\rho})$ by Theorem 3.2. In particular, $u_0 = u_T(-T) \in \Sigma$ is well-defined and $u(t,x) \in C(\mathbb{R};\Sigma) \cap L^q(\mathbb{R};W^{1,\rho})$ is the unique solution of (NLS) with the initial data u_0 such that, by use of $u \in \tilde{X}_T$, $u = \Psi_+(u)$ and Strichartz estimate,

$$\begin{aligned}\|S(-t)u(t) - u_+\|_\Sigma &= \left\| i\int_t^\infty S(-t')(|u|^{p-1}u)(t')dt' \right\|_\Sigma \\ &\leqslant Ct^{-(\frac{2p}{q}-1)}\|u\|^p_{\tilde{X}_T} \to 0 \text{ as } t \to \infty.\end{aligned}$$

The solution $u \in C(\mathbb{R};\Sigma) \cap L^q(\mathbb{R};W^{1,\rho}(\mathbb{R}^d))$ such that $\|S(-t)u(t) - u_+\|_\Sigma \to 0$ as $t \to \infty$ is unique: Indeed, if there are two solutions u_1, u_2 of (NLS) such that $\|S(-t)u_j(t) - u_+\|_\Sigma \to 0$ as $t \to \infty$, then $u_+ = u_j(0) - i\int_0^\infty S(-t')(|u_j|^{p-1}u_j)(t')dt'$ which together with the Duhamel's formula for u_j implies that (u_j)s are the unique fixed point of the nonlinear map Ψ_+.

Similarly we can define the wave operator $\Omega_- : u_- \to u_0$, $\Sigma \mapsto \Sigma$. □

[21.12.2018]

[09.01.2019]

4 Solitary waves

4.1 A minimiser problem

4.1.1 Space $H^1_r(\mathbb{R}^d)$

Let $d \geqslant 2$. Let $H^1_r(\mathbb{R}^d)$ be the set of the radial functions in $H^1(\mathbb{R}^d)$:

$$H^1_r(\mathbb{R}^d) = \{f \in H^1(\mathbb{R}^d) \mid \exists \tilde{f} : [0,\infty) \to \mathbb{C} \text{ s.t. } f(x) = \tilde{f}(r),\, r = (\sum_{j=1}^d |x_j|^2)^{\frac{1}{2}}\}.$$

[4] We write $u(t+T) = S(t+T)u_+ + i\int_t^\infty S(t+T-t')(|u|^{p-1}u)(t')dt'$ and $u(T) = S(T)u_+ + i\int_T^\infty S(T-t')(|u|^{p-1}u)(t')dt'$ and take the difference between $u(t+T)$ and $S(t)u(T)$.

It can also be viewed as the completement of the set of the radial functions in $C_0^\infty(\mathbb{R}^d)$ with respect to the H^1-norm:

$$\|f\|^2_{H^1(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |\nabla f|^2 + |f|^2 \,\mathrm{d}x \;= \omega_d \int_0^\infty (|\partial_r \tilde{f}(r)|^2 + |\tilde{f}(r)|^2) r^{d-1} dr,$$

where ω_d is the area of the unit sphere in $\mathbb{R}^d$.

Lemma 4.1 (Regularity and vanishing property of H^1_r-functions)**.** *Let $d \geqslant 2$ and $u \in H^1_r(\mathbb{R}^d)$. Then $\tilde{u} \in C^{\frac{1}{2}}((0,\infty))$ and*

$$\|r^{\frac{d-1}{2}} u\|_{L^\infty(\mathbb{R}^d)} \leqslant C\|u\|_{H^1(\mathbb{R}^d)}. \tag{4.10}$$

Proof. Let $\varphi(x) = \tilde{\varphi}(r) \in C_0^\infty(\mathbb{R}^d)$. Then

$$\tilde{\varphi}^2(r) = -2\int_r^\infty \tilde{\varphi}'(\rho)\tilde{\varphi}(\rho) d\rho,$$

and hence

$$\begin{aligned}|\tilde{\varphi}^2|(r) &\leqslant \frac{2}{r^{d-1}} \int_r^\infty |\tilde{\varphi}'\tilde{\varphi}(\rho)|\rho^{d-1} d\rho \\ &\leqslant \frac{2}{r^{d-1}} \Big(\int_r^\infty |\tilde{\varphi}'|^2 \rho^{d-1} d\rho\Big)^{\frac{1}{2}} \Big(\int_r^\infty |\tilde{\varphi}|^2 \rho^{d-1} d\rho\Big)^{\frac{1}{2}} \leqslant \frac{C}{r^{d-1}} \|\nabla\varphi\|^{\frac{1}{2}}_{L^2(\mathbb{R}^d)} \|\varphi\|^{\frac{1}{2}}_{L^2(\mathbb{R}^d)}.\end{aligned}$$

This implies (4.10) by density argument.

Similarly, let $0 < r_1 < r_2 < \infty$ and we calculate by Hölder's inequality

$$\begin{aligned}|\tilde{\varphi}(r_1) - \tilde{\varphi}(r_2)| &\leqslant \Big|\int_{r_1}^{r_2} \tilde{\varphi}' d\rho\Big| \leqslant \frac{1}{r_1^{\frac{d-1}{2}}} \int_{r_1}^{r_2} |\tilde{\varphi}'|\rho^{\frac{d-1}{2}} d\rho \\ &\leqslant \frac{C}{r_1^{\frac{d-1}{2}}} \|\nabla\varphi\|_{L^2(\mathbb{R}^d)} |r_2 - r_1|^{\frac{1}{2}},\end{aligned}$$

which implies that for any compact set $K \subset\subset (0,\infty)$, $\|\tilde{\varphi}\|_{C^{\frac{1}{2}}(\overline{K})} = \|\tilde{\varphi}\|_{L^\infty(K)} + \sup_{r_1 \neq r_2, r_1, r_2 \in K} \frac{|\tilde{\varphi}(r_2) - \tilde{\varphi}(r_1)|}{|r_2 - r_1|^{1/2}} \leqslant C(K)\|\varphi\|_{H^1}$ and hence by density argument $\|\tilde{u}\|_{C^{\frac{1}{2}}(\overline{K})} \leqslant C(K)\|u\|_{H^1}$ which implies $\tilde{u} \in C^{\frac{1}{2}}((0,\infty))$. □

Proposition 4.1 (Compact Sobolev embedding)**.** *Let $d \geqslant 2$ and 2^* as defined in Corollary 2.1. Then the Sobolev embedding $H^1_r(\mathbb{R}^d) \hookrightarrow\hookrightarrow L^q(\mathbb{R}^d)$, $q \in (2, 2^*)$ is compact.*

Proof. Let $u \in H^1_r(\mathbb{R}^d)$ and $2 < q < 2^*$. Then (4.10) implies

$$\int_{|x|\geqslant R} |u|^q \,\mathrm{d}x \leqslant \frac{\|r^{\frac{d-1}{2}} u\|^{q-2}_{L^\infty}}{R^{\frac{(q-2)(d-1)}{2}}} \int_{\mathbb{R}^d} |u|^2 \,\mathrm{d}x \leqslant CR^{-\frac{(q-2)(d-1)}{2}} \|u\|^q_{H^1(\mathbb{R}^d)}$$
$$\to 0 \text{ as } R \to \infty \text{ uniformly for the } H^1_r \text{ functions with } \|u\|_{H^1} \leqslant 1.$$

This, combined with the compact embedding $H^1(\mathbb{R}^d) \hookrightarrow\hookrightarrow L^q(\bar{B}_R)$ for any $R \in (0,\infty)$ in Theorem 2.6, implies the compact embedding $H^1_r(\mathbb{R}^d) \hookrightarrow\hookrightarrow L^q(\mathbb{R}^d)$. ☐

Remark 4.1. *We do not have the endpoint case $H^1_r(\mathbb{R}^d) \hookrightarrow\hookrightarrow L^2(\mathbb{R}^d)$ (**Exercise**.)*

4.1.2 Compact minimisation

Proposition 4.2 (Compact minimisation). *Let $d \geqslant 2$ and p be a energy-subcritical exponent: $1 < p < 2^* - 1 = \begin{cases} 1 + \frac{4}{d-2} & \text{if } d \geqslant 3 \\ \infty & \text{if } d = 2 \end{cases}$.*

Then for any $M > 0$, the minimisation problem

$$\begin{aligned} I_M &= \inf_{u \in \mathcal{A}_M} \{\|u\|^2_{H^1(\mathbb{R}^d)}\}, \\ \text{where } \mathcal{A}_M &= \Big\{u \in H^1_r(\mathbb{R}^d) \mid \int_{\mathbb{R}^d} |u|^{p+1} \,\mathrm{d}x = M\Big\}, \end{aligned} \tag{4.11}$$

has a solution $u \in \mathcal{A}_M$ and $I_M > 0$.

Proof. Since by Sobolev's embedding $H^1(\mathbb{R}^d) \hookrightarrow L^{p+1}(\mathbb{R}^d)$, $2 < p+1 < 2^*$: $\|u\|_{H^1(\mathbb{R}^d)} \geqslant C^{-1}\|u\|_{L^{p+1}(\mathbb{R}^d)} = C^{-1}M^{1/(p+1)}$ if $u \in \mathcal{A}_M$, we can take a minimising sequence $(u_n)_n$ in $\mathcal{A}_M$ such that

$$\|u_n\|^2_{H^1(\mathbb{R}^d)} \to I_M \geqslant C^{-2} M^{\frac{2}{p+1}} > 0.$$

Since $(u_n)_n$ are bounded in $H^1_r(\mathbb{R}^d)$, by Proposition 4.1 there exists a subsequence (still denoted by $(u_n)_n$) and $u \in H^1_r(\mathbb{R}^d)$ such that

$$u_n \to u \text{ in } L^{p+1}(\mathbb{R}^d), \quad u_n \rightharpoonup u \text{ in } H^1(\mathbb{R}^d).$$

Therefore

$$\begin{aligned} &\int_{\mathbb{R}^d} |u|^{p+1} \,\mathrm{d}x = \lim_{n\to\infty} \int_{\mathbb{R}^d} |u_n|^{p+1} \,\mathrm{d}x = M \text{ and thus } u \in \mathcal{A}_M, \\ &\|u\|^2_{H^1(\mathbb{R}^d)} \leqslant \liminf_{n\to\infty} \|u_n\|^2_{H^1(\mathbb{R}^d)} = I_M, \end{aligned}$$

and hence $u \in \mathcal{A}_M$ is the minimiser of (4.11). ☐

Lemma 4.2 (Positivity)**.** *If $u \in \mathcal{A}_M$, then $|u| \in \mathcal{A}_M$, $\||u|\|_{H^1(\mathbb{R}^d)} \leqslant \|u\|_{H^1(\mathbb{R}^d)}$. If $u \in \mathcal{A}_M$ is a minimiser of (4.11), then so is $|u|$ such that $\||u|\|_{H^1(\mathbb{R}^d)} = \|u\|_{H^1(\mathbb{R}^d)}$, and if furthermore $|u| > 0$, then $u = |u|e^{i\gamma}$ for some $\gamma \in \mathbb{R}$.*

Proof. The lemma follows from the following claim that if $u \in H^1(\mathbb{R}^d)$, then

$$\int_{\mathbb{R}^d} |\nabla u|^2 \, \mathrm{d}x \geqslant \int_{\mathbb{R}^d} |\nabla |u||^2 \, \mathrm{d}x$$

and if $|u| > 0$, then the above equality holds if and only if $u = |u|e^{i\gamma}$ for some $\gamma \in \mathbb{R}$. (**Exercise** Prove this claim.) □

4.1.3 Euler-Lagrangian equation

Proposition 4.3. *Let $u \geqslant 0$ be the minimiser of (4.11). Then there exists $\lambda \in \mathbb{R}$ such that*

$$-\Delta u + u = \lambda u^p. \tag{4.12}$$

The λ in (4.12) is indeed a positive constant independent on u: $\lambda = I_M/M$.

Proof. **Step 1 Differentiation**

Let $t \in \mathbb{R}$ and $h \in C_0^\infty(\mathbb{R}^d;\mathbb{R})$ a radial, real-valued function. Then (**Exercise**)

$$\Big|\int_{\mathbb{R}^d} \Big(|u + th|^{p+1} - u^{p+1} - (p+1)thu^p\Big) \, \mathrm{d}x \Big| \leqslant C \int_{\mathbb{R}^d} t^2 h^2 u^{p-1} + t^{p+1}|h|^{p+1} \, \mathrm{d}x \, .$$

Hence

$$\int_{\mathbb{R}^d} |u + th|^{p+1} \, \mathrm{d}x = \int_{\mathbb{R}^d} u^{p+1} \, \mathrm{d}x + (p+1)t \int_{\mathbb{R}^d} hu^p \, \mathrm{d}x + o(t) \text{ as } t \to 0.$$

Let h be chosen such that $\int_{\mathbb{R}^d} hu^p \, \mathrm{d}x = 0$, then since $u \in \mathcal{A}_M$, we have

$$\int_{\mathbb{R}^d} |u + th|^{p+1} \, \mathrm{d}x = M + o(t).$$

Let $v_t = \frac{M^{\frac{1}{p+1}}}{\|u+th\|_{L^{p+1}}}(u + th)$, then $\|v_t\|_{L^{p+1}} = M^{\frac{1}{p+1}}$, $v_t = (u + th)(1 + o(t))$ and

$$\begin{aligned}\|v_t\|_{H^1}^2 &= (1 + o(t))\Big(\|u\|_{H^1}^2 + 2t \int_{\mathbb{R}^d} (uh + \nabla u \cdot \nabla h) \, \mathrm{d}x + t^2 \|h\|_{H^1}^2\Big) \\ &= \|u\|_{H^1}^2 + 2t \int_{\mathbb{R}^d} (uh + \nabla u \cdot \nabla h) \, \mathrm{d}x + o(t) \text{ as } t \to 0.\end{aligned}$$

Since $u \geqslant 0$ is the minimiser, $\|v_t\|_{H^1} \geqslant \|u\|_{H^1}$ for any $t \in \mathbb{R}$ and hence

$$\int_{\mathbb{R}^d} (uh + \nabla u \cdot \nabla h) \, \mathrm{d}x = 0 \text{ for } h \in C_0^\infty(\mathbb{R}^d; \mathbb{R}) \text{ radial,}$$
$$\text{whenever } \int_{\mathbb{R}^d} hu^p \, \mathrm{d}x = 0.$$

Step 2 Lagrangian multiplier

Let L_1, L_2 be the two linear forms on the Hilbert space H_r^1 defined by

$$L_1(h) = \int_{\mathbb{R}^d} hu^p \, \mathrm{d}x \,, \quad L_2(h) = \int_{\mathbb{R}^d} (uh + \nabla u \cdot \nabla h) \, \mathrm{d}x \,.$$

Then $\operatorname{Ker} L_1 \subset \operatorname{Ker} L_2$. Let $h \in H_r^1$ with $L_1(h) \neq 0$. Then any $a \in H_r^1$ can be written as

$$a = \frac{L_1(a)}{L_1(h)} h + b \text{ with } b = a - \frac{L_1(a)}{L_1(h)} h \in \operatorname{Ker} L_1 \subset \operatorname{Ker} L_2.$$

Hence $L_2(a) = \frac{L_1(a)}{L_1(h)} L_2(h) = (\frac{L_2(h)}{L_1(h)}) L_1(a)$ for any $a \in H_r^1$. This implies (4.12).

Step 3 Lagrange multiplicator

We test (4.12) by $\bar{u} = u \geqslant 0$ to arrive at

$$I_M = \int_{\mathbb{R}^d} |\nabla u|^2 + |u|^2 \, \mathrm{d}x = \lambda \int_{\mathbb{R}^d} u^{p+1} \, \mathrm{d}x = \lambda M,$$

which implies $\lambda = I_M / M > 0$. □

[09.01.2019]

[11.01.2019]

4.1.4 Regularity and decay property

We take $v = \lambda^{\frac{1}{p-1}} u$ in (4.12) such that v satisfies the following renormalised equation with $1 < p < 2^* - 1$

$$\Delta v - v + v^p = 0, \quad v \geqslant 0, \quad v \in H_r^1. \tag{4.13}$$

Proposition 4.4 (Regularity and decay)**.** *Let $v(x) = \tilde{v}(r) \neq 0$ solves (4.13), then $v \in W^{3,q}(\mathbb{R}^d)$, $\forall q \in [2, \infty)$ such that*

$$v \in C^2(\mathbb{R}^d), \quad |D^\beta v(x)| \to 0 \textit{ as } |x| \to \infty, \forall |\beta| \leqslant 2,$$
$$\exists \varepsilon > 0 \textit{ s.t. } e^{\varepsilon |x|}(|v| + |\nabla v|) \in L^\infty(\mathbb{R}^d),$$

and $\tilde{v}$ solves the ODE

$$\tilde{v}'' + \frac{d-1}{r}\tilde{v}' = \tilde{v} - \tilde{v}^p, \quad \tilde{v}(0) = a,\ \tilde{v}'(0) = 0, \tag{4.14}$$

for some $a > 0$.

Proof. **Step 1 Regularity by iteration**

We will use freely the following fact which we admit here without proof: If $v \in L^q(\mathbb{R}^d)$, $1 < q < \infty$, then $(1-\Delta)^{-1} v \in W^{2,q}$.

By view of $v \in H^1_r(\mathbb{R}^d) \hookrightarrow L^{q_0}(\mathbb{R}^d)$, $q_0 = p+1$ and the Sobolev embedding

$$W^{2,\frac{q_j}{p}}(\mathbb{R}^d) \hookrightarrow \begin{cases} L^{q_{j+1}}(\mathbb{R}^d) \text{ if } 2 - \frac{dp}{q_j} = -\frac{d}{q_{j+1}} < 0 \\ L^q(\mathbb{R}^d),\ \forall q \in [\frac{q_j}{p}, \infty),\ \text{ if } 2 - \frac{dp}{q_j} = 0 \\ L^\infty(\mathbb{R}^d) \text{ if } 2 - \frac{dp}{q_j} > 0. \end{cases}$$

we have $v \in L^\infty(\mathbb{R}^d)$. Indeed, as $v^p \in L^{\frac{q_0}{p}}$, $v = (1-\Delta)^{-1} v \in W^{2,\frac{q_0}{p}}$,

- if $2 - \frac{dp}{q_0} > 0$, then by Sobolev embedding $v \in W^{2,\frac{q_0}{p}}(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$.
- if $2 - \frac{dp}{q_0} < 0$, then by Sobolev embedding $v \in W^{2,\frac{q_0}{p}}(\mathbb{R}^d) \hookrightarrow L^{q_1}(\mathbb{R}^d)$ and hence $v \in W^{2,\frac{q_1}{p}}(\mathbb{R}^d)$. If $2 - \frac{dp}{q_1} > 0$, then we are done. If not, we can continue the procedure such that there exists k with $2 - \frac{dp}{q_k} \geqslant 0$ and $2 - \frac{dp}{q_{k-1}} < 0$: This is possible since

$$\frac{1}{q_{j+1}} = -\frac{2}{d} + \frac{p}{q_j}$$
$$\Rightarrow \frac{1}{q_{j+1}} - \frac{1}{q_j} = p^j\Big(\frac{p-1}{q_0} - \frac{2}{d}\Big), \text{ with } \frac{p-1}{p+1} - \frac{2}{d} < 0 \text{ if } p < 2^* - 1.$$

- if $2 - \frac{dp}{q_k} = 0$ for some $k \in \mathbb{N}$, then $v \in L^q(\mathbb{R}^d)$ for any $q \in [2, \infty)$ and we choose $q \gg 1$ such that $2 - \frac{dp}{q} > 0$.

Therefore $v^p \in L^{\frac{p+1}{p}}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and hence $v \in W^{2,q}(\mathbb{R}^d)$ for all $q \in [2, \infty)$ and thus $v^p \in W^{1,q}(\mathbb{R}^d)$ for all $q \in [2, \infty)$ which implies correspondingly $\nabla v \in W^{2,q}(\mathbb{R}^d)$ for all $q \in [2, \infty)$. Thus $v \in W^{3,q}(\mathbb{R}^d)$ for all $q \in [2, \infty)$, which implies

- $v \in C^{2,\alpha}(\mathbb{R}^d)$ for any $\alpha \in (0,1)$, by Sobolev embedding;
- $\forall |\beta| \leqslant 2$, $D^\beta v \in H^1_r(\mathbb{R}^d)$ and hence $|D^\beta v(x)| \to 0$ as $|x| \to \infty$.

Hence the equation (4.13) for $v(x)$ implies the equation (4.14) for $\tilde{v}(r)$ in the classical sense, and furthermore, $\tilde{v}' = \frac{r}{d-1}(\tilde{v} - \tilde{v}^p - \tilde{v}'')$ is uniformly bounded such that $\tilde{v}'(r) \to 0$ as $r \to 0$ and thus $\tilde{v}(0) = a > 0$ since if $a = 0$ then $\tilde{v} = 0$.

Step 2 Decay property

Let $\theta_\varepsilon = e^{\frac{|x|}{1+\varepsilon|x|}}$, $\varepsilon > 0$ be a bounded, Lipschitz continuous function with $|\nabla\theta_\varepsilon|^2 \leqslant \theta_\varepsilon^2$, a.e. We test the equation (4.13) by $\theta_\varepsilon v$ to get (**Exercise**)

$$\int_{\mathbb{R}^d} \theta_\varepsilon v^2 \,\mathrm{d}x < \infty,$$

which implies $\int_{\mathbb{R}^d} e^{|x|} v^2 \,\mathrm{d}x < \infty$ as $\varepsilon \to 0$. Since v is globally Lipschitz continuous, $e^{|x|} v^{d+2}$ is uniformly bounded. Similarly we apply ∂_{x_j} to the equation (4.13) and test it by $\theta_\varepsilon \partial_{x_j} v$ to arrive at $\int_{\mathbb{R}^d} e^{|x|} |\nabla v|^2 \,\mathrm{d}x < \infty$.

□

Remark 4.2. *It is easy to show the regularity away from the origin by Lemma 4.1. Indeed, let $w = \chi v$, where $\chi \in C_0^\infty$ is a radial function with the compact support away from zero and v satisfies* (4.13)*. Then w satisfies*

$$\Delta w - w = f, \quad f = -\chi v^p + 2\nabla\chi \cdot \nabla v + v\Delta\chi.$$

Since $v \in L^\infty$ on $\operatorname{Supp}\chi$ by virtue of (4.10)*, $f \in L^2(\mathbb{R}^d)$ and hence $\hat{w}(\xi) = -\frac{\hat{f}}{1+|\xi|^2}$, that is, $w \in H_r^2(\mathbb{R}^d)$. Thus $\partial_r \tilde{w} \in C^{\frac{1}{2}}((0,\infty))$ by Lemma 4.1 and $w(x) = \tilde{w}(r) \in C^1(\mathbb{R}^d \backslash \{0\})$, $v(x) \in C^1(\mathbb{R}^d \backslash \{0\})$. Now consider the equation for $\partial_r \tilde{w}$:*

$$(\Delta - 1)\partial_r \tilde{w} = \partial_r f \in L^2(\mathbb{R}^d),$$

and the same argument as before implies $v(x) \in C^2(\mathbb{R}^d \backslash \{0\})$.

4.1.5 Classification of minimisers

We have the following uniqueness result for the nonnegative solution which decays at infinity of the equation (4.14) proved by [Kwong 1987]:

Proposition 4.5 (Uniqueness)**.** *There exists a unique $a > 0$ such that the solution $\tilde{v}$ of the ODE* (4.14) *satisfying*

$$\tilde{v}(r) \geqslant 0, \forall r \geqslant 0 \text{ and } \tilde{v}(r) \to 0 \text{ as } r \to \infty.$$

Furthermore, $\tilde{v}(r) > 0$ for all $r \geqslant 0$ and we denote the solution to be $Q(r)$: the fundamental solution of (4.14)*.*

We do not give a proof here and interested readers can refer to Appendix B of Tao's book.

Let $u \in \mathcal{A}_M$ be a minimiser of (4.11), then $|u| \in \mathcal{A}_M$ is also a minimiser by Lemma 4.2. Thus by Propositions 4.3 and 4.4, the nonnegative function $v = \lambda^{\frac{1}{p-1}}|u| \in H^1_r$ satisfies (4.13) and the nonnegative function $\tilde{v}(r) = v(x)$ satisfies (4.14) and $\tilde{v}(r) \to 0$ as $r \to \infty$. Hence by Proposition 4.5, $v(x) = \tilde{v}(r) = Q(r) > 0$ and thus $|u| > 0$. Since $u, |u| > 0$ are two minimisers such that $\|\nabla|u|\|_{L^2(\mathbb{R}^d)} = \|\nabla u\|_{L^2(\mathbb{R}^d)}$, there exists $\gamma \in \mathbb{R}$ such that

$$u = |u|e^{i\gamma} = \lambda^{-\frac{1}{p-1}} v e^{i\gamma} = (\frac{I_M}{M})^{-\frac{1}{p-1}} Q(r) e^{i\gamma}.$$

Therefore we have obtained

Theorem 4.1 (Classification of minimisers). *Let $M > 0$, $d \geqslant 2$, $1 < p < 2^* - 1$, then the minimisation problem* (4.11)

$$I_M = \inf_{u \in \mathcal{A}_M} \{\|u\|^2_{H^1}\}, \quad \mathcal{A}_M = \{u \in H^1_r(\mathbb{R}^d) \mid \int_{\mathbb{R}^d} |u|^{p+1}\,\mathrm{d}x = M\}$$

has a family of minimisers

$$e^{i\gamma}(\frac{M}{I_M})^{\frac{1}{p-1}} Q(r), \quad \gamma \in \mathbb{R},$$

where $Q > 0$ is the unique fundamental state of the equation (4.13).

A similar proof in [Weinstein '1983 CMP] explains the optimal constant in the Gagliardo-Nirenberg's inequality

$$\inf_{f \in H^1} \frac{\|f\|^{\frac{4}{d}}_{L^2(\mathbb{R}^d)} \|\nabla f\|^2_{L^2(\mathbb{R}^d)}}{\|f\|^{2+\frac{4}{d}}_{L^{2+\frac{4}{d}}(\mathbb{R}^d)}} = \frac{\|Q\|^{\frac{4}{d}}_{L^2(\mathbb{R}^d)}}{1 + \frac{2}{d}},$$

4.2 Concentration compactness

Lemma 4.3 (Concentration-Compactness). *Let $(u_n)_{n \geqslant 1}$ be a bounded sequence in $H^1(\mathbb{R}^d)$ with $\|u_n\|^2_{L^2(\mathbb{R}^d)} = M > 0$. Then there exists a subsequence (u_{n_k}) such that one the following properties holds:*

(i) Compactness: There exists a sequence (y_k) in $\mathbb{R}^d$ such that

$$\forall q \in [2, 2^*), \quad u_{n_k}(\cdot - y_k) \to u \text{ in } L^q(\mathbb{R}^d) \text{ as } k \to \infty;$$

(ii) Evanescence: $\forall q \in (2, 2^*)$*,* $u_{n_k} \to 0$ *in* $L^q(\mathbb{R}^d)$ *as* $k \to \infty$*;*

(iii) Dichotomy: There exist two bounded sequences (v_k)*,* (w_k) *with compact support in* $H^1(\mathbb{R}^d)$ *and* $\alpha \in (0, 1)$*, such that*

$$\begin{aligned}
&\operatorname{Supp} v_k \cap \operatorname{Supp} w_k = \{\}, \quad d(\operatorname{Supp} v_k, \operatorname{Supp} w_k) \to \infty \text{ as } k \to \infty,\\
&\|v_k\|^2_{L^2(\mathbb{R}^d)} \to \alpha M, \quad \|w_k\|^2_{L^2(\mathbb{R}^d)} \to (1-\alpha)M, \text{ as } k \to \infty,\\
&\forall q \in [2, 2^*), \quad \|u_{n_k}\|^q_{L^q} - \|v_k\|^q_{L^q} - \|w_k\|^q_{L^q} \to 0, \text{ as } k \to \infty,\\
&\liminf_{k\to\infty}(\|\nabla u_{n_k}\|^2_{L^2} - \|\nabla v_k\|^2_{L^2} - \|\nabla w_k\|^2_{L^2}) \geqslant 0.
\end{aligned}$$

Proof. **Step 1 Concentration functions**

Let $\rho_n : [0, \infty) \mapsto [0, M]$ be the concentration function of u_n:

$$\rho_n(R) = \sup_{y\in\mathbb{R}^d} \int_{B(y,R)} |u_n(x)|^2 \, \mathrm{d}x \, ,$$

with the following properties:

- Monotonicity: $\forall n$, $\rho_n(R)$ increases to M as R increases to ∞;
- Concentration point: $\forall R$, the map $y \mapsto \int_{B(y,R)} |u|^2$ is continuous and tends to zero as $|y| \to \infty$, and hence the concentration point exists:

  $$\forall R > 0, \quad \forall n \geqslant 0, \quad \exists y_n = y_n(R) \in \mathbb{R}^d \text{ s.t. } \rho_n(R) = \int_{B(y_n,R)} |u_n|^2 \, \mathrm{d}x \, ;$$

- Uniform Hölder continuity: There exist $C, \beta > 0$ (independent on n) such that

  $$\forall R_1, R_2 > 0, \quad \forall n \geqslant 0, \quad |\rho_n(R_1) - \rho_n(R_2)| \leqslant C|R_2^d - R_1^d|^\beta.$$

  Indeed, suppose without loss of generality $R_1 \leqslant R_2$, then

  $$\begin{aligned}
  &|\rho_n(R_1) - \rho_n(R_2)| = \int_{B(y_n^2,R_2)} |u_n|^2 \, \mathrm{d}x - \int_{B(y_n^1,R_1)} |u_n|^2 \, \mathrm{d}x\\
  &= \Big(\int_{B(y_n^2,R_2)} - \int_{B(y_n^2,R_1)}\Big)|u_n|^2 \, \mathrm{d}x + \Big(\int_{B(y_n^2,R_1)} - \int_{B(y_n^1,R_1)}\Big)|u_n|^2 \, \mathrm{d}x\\
  &\leqslant \int_{R_1\leqslant|x-y_n^2|\leqslant R_2} |u_n|^2 \, \mathrm{d}x \leqslant C\Big(\int_{R_1\leqslant|x-y_n^2|\leqslant R_2} |u_n|^{2^*} \, \mathrm{d}x\Big)^{\frac{2}{2^*}} (R_2^d - R_1^d)^{1-\frac{2}{2^*}}\\
  &\leqslant C\|u_n\|^2_{H^1(\mathbb{R}^d)}(R_2^d - R_1^d)^{1-\frac{2}{2^*}} \text{ if } 2^* < \infty \text{ i.e. } d \geqslant 3 \text{ s.t. } H^1(\mathbb{R}^d) \hookrightarrow L^{2^*}(\mathbb{R}^d).
  \end{aligned}$$

  By Sobolev embedding $H^1(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d)$ for all $p \in [2, \infty)$ if $d = 1, 2$, then the above argument also holds with 2^* replaced by any $p > 2$.

By Arzela-Ascoli's Theorem, the uniform Hölder continuity of the sequence (ρ_n) above implies the existence of a subsequence (ρ_{n_k}) and a Hölder continuous monotone function $\rho(R)$ such that

$$\forall R > 0, \quad \lim_{k\to\infty} \rho_{n_k}(R) = \rho(R).$$

Let $m = \lim_{R\to\infty} \rho(R) \leqslant M$. Then (**Exercise**) there exists a sequence $R_k \to \infty$ such that

$$m = \lim_{k\to\infty} \rho_{n_k}(R_k) = \lim_{k\to\infty} \rho_{n_k}(\frac{R_k}{2}) = \lim_{R\to\infty} \rho(R).$$

[11.01.2019]

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Step 2 Case $m = 0$: Evanescence

Since $\rho : [0,\infty) \mapsto [0,m]$ is an increasing function, then $\rho = 0$ if $m = 0$. In particular

$$\lim_{k\to\infty} \rho_{n_k}(1) = \rho(1) = 0 = \lim_{k\to\infty} \sup{}_{y\in\mathbb{R}^d} \int_{B(y,1)} |u_{n_k}|^2 \,\mathrm{d}x\,.$$

This uniformly local strong convergence in $L^2(\mathbb{R}^d)$ implies the strong convergence in $L^q(\mathbb{R}^d)$, $q \in (2,2^*)$: $u_{n_k} \to 0$ in $L^q(\mathbb{R}^d)$. Indeed, by use of the unity partition (Q_j) (such that each Q_j is contained in a ball of radius 1), we have the following version of Gagliardo-Nirenberg's inequality:

$$\begin{aligned}\int_{\mathbb{R}^d} |u|^{2+\frac{4}{d}} \,\mathrm{d}x &= \sum_{j\geqslant 1} \|u\|_{L^{2+\frac{4}{d}}(Q_j)}^{2+\frac{4}{d}} \leqslant C \sum_{j\geqslant 1} \|u\|_{L^2(Q_j)}^{\frac{4}{d}} \|\nabla u\|_{L^2(Q_j)}^2 \\ \leqslant C\sup{}_{j\geqslant 1} \|u\|_{L^2(Q_j)}^{\frac{4}{d}} &\sum_{j\geqslant 1} \|\nabla u\|_{L^2(Q_j)}^2 \leqslant C\Big(\sup{}_{j\geqslant 1} \|u\|_{L^2(Q_j)}^2\Big)^{\frac{2}{d}} \|\nabla u\|_{L^2}^2,\end{aligned}$$

for $d \geqslant 3$, and for $d = 1, 2$, we can take use of $\|u\|_{L^4} \leqslant C\|u\|_{L^2}^{\frac{1}{2}} \|u\|_{H^1}^{\frac{1}{2}}$. We arrive at $u_{n_k} \to 0$ in $L^{2+\frac{4}{d}}(\mathbb{R}^d)$ or $L^4(\mathbb{R}^d)$ and the interpolation in the Lebesgue spaces implies $u_{n_k} \to 0$ in L^q, $q \in (2,2^*)$.

Step 3 Case $m = M$: Compactness

For any $R > 0$, let $y_k(R)$ be such that $\rho_{n_k}(R) = \int_{B(y_k(R),R)} |u_{n_k}|^2 \,\mathrm{d}x\,$. There exist R_0, k_0 such that

$$\rho_{n_k}(R_0) = \int_{B(y_k(R_0),R_0)} |u_{n_k}|^2 \,\mathrm{d}x > \frac{M}{2}, \quad \forall k \geqslant k_0,$$

and for any $\varepsilon > 0$, then there exist $R_\varepsilon, k_\varepsilon \geqslant k_0$ such that

$$\rho_{n_k}(R_\varepsilon) = \int_{B(y_k(R_\varepsilon),R_\varepsilon)} |u_{n_k}|^2 \,\mathrm{d}x \; > M - \varepsilon, \quad \forall k \geqslant k_\varepsilon.$$

Since u_{n_k} has the total mass M, the two balls $B(y_k(R_0), R_0) \cap B(y_k(R_\varepsilon), R_\varepsilon) \neq \{\}$ and hence there exists $R_{0\varepsilon}$ such that

$$\int_{B(y_k(R_0),R_{0\varepsilon})} |u_{n_k}|^2 \,\mathrm{d}x \; > M - \varepsilon, \quad \forall k \geqslant k_\varepsilon.$$

We may assume that the above holds true for all k by choosing a possibly larger $R_{0\varepsilon}$ and hence $v_k = u_{n_k}(\cdot - y_k(R_0))$ satisfies

$$\forall \varepsilon > 0, \quad \exists R_{0\varepsilon} \text{ s.t. } \forall k \geqslant 1, \quad \int_{|x| \geqslant R_{0\varepsilon}} |v_k|^2 \,\mathrm{d}x \; < \varepsilon.$$

By virtue of the compact embedding $H^1(\mathbb{R}^d) \hookrightarrow\hookrightarrow L^2(B(0, R_{0\varepsilon}))$, $v_k \to u$ in $L^q(\mathbb{R}^d)$, $q \in [2, 2^*)$.

Step 4 Case $0 < m < M$: Dichotomy

We decompose u_{n_k} as

$$\begin{aligned} u_{n_k} &= u_{n_k} \mathbf{1}_{|y - y_k(\frac{R_k}{2})| \leqslant \frac{R_k}{2}} + u_{n_k} \mathbf{1}_{|y - y_k(\frac{R_k}{2})| \geqslant R_k} + u_{n_k} \mathbf{1}_{\frac{R_k}{2} < |y - y_k(\frac{R_k}{2})| < R_k} \\ &:= v_k + w_k + z_k, \end{aligned}$$

then

$$\begin{aligned} \int_{\mathbb{R}^d} |z_k|^2 \,\mathrm{d}x \; &= \Big(\int_{B(y_k(\frac{R_k}{2}), R_k)} - \int_{B(y_k(\frac{R_k}{2}), \frac{R_k}{2})} \Big) |u_{n_k}|^2 \,\mathrm{d}x \\ &\leqslant \rho_{n_k}(R_k) - \rho_{n_k}(\frac{R_k}{2}) \to 0 \text{ as } k \to \infty. \end{aligned}$$

We then replace the characterised functions $\mathbf{1}_{|y - y_k(\frac{R_k}{2})| \leqslant \frac{R_k}{2}}, \mathbf{1}_{|y - y_k(\frac{R_k}{2})| \geqslant R_k}$ by regular cutoff functions θ_k, φ_k with compact supports and $\sup_k \|\nabla\theta_k\|_{L^\infty}, \sup_k \|\nabla\varphi_k\|_{L^\infty} \leqslant 4R_k^{-1}$ such that v_k, w_k are compactly supported functions with $\|v_k\|_{L^2}^2 \to m$, $\|w_k\|_{L^2}^2 \to M - m$. Since

$$\begin{aligned} |\nabla u_{n_k}|^2 - |\nabla v_k|^2 - |\nabla w_k|^2 &= |\nabla u_{n_k}|^2 (1 - |\theta_k|^2 - |\varphi_k|^2) \\ &\quad - |u_{n_k}|^2 (|\nabla\theta_k|^2 + |\nabla\varphi_k|^2) - \mathrm{Re}\,(\overline{u_{n_k}} \nabla u_{n_k}) \cdot \nabla(\theta_k^2 + \varphi_k^2) \\ &\geqslant -16 |u_{n_k}|^2 (R_k)^{-2} - 8 |u_{n_k}| |\nabla u_{n_k}| (R_k)^{-1}, \end{aligned}$$

we have $\liminf_{k\to\infty}\int_{\mathbb{R}^d}|\nabla u_{n_k}|^2-|\nabla v_k|^2-|\nabla w_k|^2\,\mathrm{d}x \geqslant 0$. Hence by virtue of $z_k\to 0$ in $L^q(\mathbb{R}^d)$, $q\in[2,2^*)$,

$$\begin{aligned}\int_{\mathbb{R}^d}\big||u_{n_k}|^q-|v_k|^q-|w_k|^q\big|\,\mathrm{d}x &\leqslant C\int_{\mathbb{R}^d}|u_{n_k}|^{q-1}|z_k|\,\mathrm{d}x\\ &\leqslant C\|u_{n_k}\|_{L^q}^{q-1}\|z_k\|_{L^q}\to 0 \text{ as } k\to\infty.\end{aligned}$$

□

Exercise. Give three examples of bounded H^1 sequences such that compactness/evanescence/dichotomy hold respectively.

Remark 4.3. *We can repeat the lemma (established by P.-L. Lions 1983' and we follow the proof in Cazenave 2004') to derive the decomposition profile of a sequence (u_n) in $H^s(\mathbb{R}^d)$ established by P. Gérard 1998', which describes the defect of the compactness of Sobolev embeddings up to extraction:*
Let (u_n) be a bounded sequence of $H^s(\mathbb{R}^d)$, $0<s<\frac{d}{2}$ and $\frac{d}{2}-s=\frac{d}{p}$. Then there exist a sequence of scales and cores $(\lambda_n^{(j)},x_n^{(j)})_{(j,n)\in\mathbb{N}^2}$ in the sense that

$$j\neq k\Rightarrow \text{ either } \lim_{n\to\infty}\Big|\log\big(\frac{\lambda_n^{(j)}}{\lambda_n^{(k)}}\big)\Big|=\infty \text{ or } \lim_{n\to\infty}\frac{|x_n^{(j)}-x_n^{(k)}|}{\lambda_n^{(j)}}=\infty,$$

a sequence (φ_j) in $H^s(\mathbb{R}^d)$ and a sequence $(r_n^{(j)})$ of functions such that

$$\begin{aligned}&\forall J\in\mathbb{N},\quad u_{\phi(n)}(x)=\sum_{j=0}^{J}\frac{1}{(\lambda_n^{(j)})^{\frac{d}{2}-s}}\varphi_j\big(\frac{x-x_n^{(j)}}{\lambda_n^{(j)}}\big)+r_n^{(J)}(x),\\ &\lim_{J\to\infty}\limsup_{n\to\infty}\|r_n^{(J)}\|_{L^p}=0,\\ &\forall J\in\mathbb{N},\quad \lim_{n\to\infty}\Big(\|u_{\phi(n)}\|_{H^s}^2-\sum_{j=0}^{J}\|\varphi_j\|_{H^s}^2-\|r_n^{(J)}\|_{H^s}^2\Big)=0.\end{aligned}$$

We also have a version of the critical case, e.g. the case $d=2$, $s=1$, $p\in(2,2^*)$, $\lambda_n^{(j)}=1$ considered by Hmidi-Keraani 2005.

4.3 Orbital stability

4.3.1 A second minimisation problem

Theorem 4.2. *Let $M>0$, $1<p<1+\frac{4}{d}$ i.e. $s_c=\frac{d}{2}-\frac{2}{p-1}<0$ and let Q be the fundamental state in Theorem 4.1. Then the minimisation problem*

$$J_M=\inf\{E(u)\,|\,u\in H^1(\mathbb{R}^d),\ \|u\|_{L^2(\mathbb{R}^d)}^2=M\} \tag{4.15}$$

is achieved by the following family of functions

$$Q_\mu(x-x_0)e^{i\gamma_0}, \quad x_0 \in \mathbb{R}^d, \quad \gamma_0 \in \mathbb{R},$$

where $E(u) = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla u|^2 \,\mathrm{d}x - \frac{1}{p+1}\int_{\mathbb{R}^d} |u|^{p+1} \,\mathrm{d}x$ *is the energy functional defined in* (1.10), $Q_\mu = \mu^{\frac{2}{p-1}} Q(\mu x)$ *and* $\mu = \mu(M) = (\frac{M}{\|Q\|_{L^2}^2})^{-\frac{1}{2s_c}}$. *Furthermore, all the minimising sequences are relatively compact in* $H^1(\mathbb{R}^d)$ *up to translation and rotation: For the sequence* (u_n) *in* $H^1(\mathbb{R}^d)$ *such that*

$$\|u_n\|_{L^2}^2 \to M, \quad E(u_n) \to J_M,$$

there exist $(x_n) \subset \mathbb{R}^d$, $(\gamma_n) \subset \mathbb{R}$ *and a subsequence* $(\phi(n))$ *such that*

$$u_{\phi(n)}(\cdot - x_{\phi(n)})e^{i\gamma_{\phi(n)}} \to Q_\mu \text{ in } H^1(\mathbb{R}^d).$$

Sketchy proof. **Step 1 Properties of J_M, $M > 0$**

We have the following properties for J_M:

- J_M has a lower bound: $J_M > -\infty$.

  Indeed, recall the Gagliardo-Nirenberg's inequality in Corollary 2.1:

  $$\|u\|_{L^{p+1}(\mathbb{R}^d)}^{p+1} \leqslant C\|\nabla u\|_{L^2(\mathbb{R}^d)}^{2a}\|u\|_{L^2(\mathbb{R}^d)}^{2b}, \tag{4.16}$$
  $$a = \frac{d(p-1)}{4}, \quad b = \frac{d+2}{4} - \frac{(d-2)p}{4}.$$

  We hence have

  $$E(u) \geqslant \frac{1}{2}\|\nabla u\|_{L^2(\mathbb{R}^d)}^2 - C\|\nabla u\|_{L^2(\mathbb{R}^d)}^{2a}\|u\|_{L^2(\mathbb{R}^d)}^{2b},$$
  $$\text{with } a < 1 \text{ since } p < 1 + \frac{4}{d}.$$

  As $\|u\|_{L^2(\mathbb{R}^d)}^2 = M$, we derive $J_M > -\infty$ by Young's inequality.

- J_M has a negative upper bound: $J_M < 0$.

  Indeed, fix some $u \in H^1(\mathbb{R}^d)$ with $\|u\|_{L^2(\mathbb{R}^d)}^2 = M$. Then the rescaled function $u^\lambda(x) = \lambda^{\frac{d}{2}}u(\lambda x)$, $\lambda > 0$ satisfies $\|u^\lambda\|_{L^2(\mathbb{R}^d)}^2 = \|u\|_{L^2(\mathbb{R}^d)}^2 = M$ and

  $$E(u^\lambda) = \lambda^2\Big[\frac{1}{2}\int_{\mathbb{R}^d} |\nabla u|^2 \,\mathrm{d}x - \frac{1}{(p+1)\lambda^{(p-1)|s_c|}}\int_{\mathbb{R}^d} |u|^{p+1} \,\mathrm{d}x\Big],$$

  and hence $E(u^\lambda) < 0$ for $\lambda > 0$ small enough.

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- J_M is homogeneous in M: $J_M = M^{\frac{1-s_c}{|s_c|}} J_1$.

  Indeed, for any $u \in H^1(\mathbb{R}^d)$, the rescaled function $u_\lambda(x) = \lambda^{\frac{2}{p-1}} u(\lambda x)$, $\lambda > 0$ satisfies $\|u_\lambda\|^2_{L^2(\mathbb{R}^d)} = \lambda^{-2s_c}\|u\|^2_{L^2(\mathbb{R}^d)} = \lambda^{-2s_c} M$ and

  $$E(u_\lambda) = \lambda^{2(1-s_c)} E(u),$$

  and hence $J_{\lambda^{-2s_c}M} = \lambda^{2(1-s_c)} J_M$ and we can choose in particular $\lambda = M^{\frac{1}{2s_c}}$.

Step 2 Existence of the minimiser

Let (u_n) be a minimizing sequence and we are going to show its compactness (up to extraction of subsequence and translation) by Lemma 4.3. We first have the following facts:

- Any subsequence is not evanescent.

  Indeed, suppose by contradiction that $u_{n_k} \to 0$ in $L^{p+1}(\mathbb{R}^d)$ as $k \to \infty$, then

  $$\begin{aligned} J_M = \lim_{k\to\infty} E(u_{n_k}) &= \lim_{k\to\infty} \Big[\frac{1}{2}\int_{\mathbb{R}^d} |\nabla u_{n_k}|^2 \, \mathrm{d}x - \frac{1}{p+1}\int_{\mathbb{R}^d} |u_{n_k}|^{p+1} \, \mathrm{d}x\Big] \\ &= \lim_{k\to\infty} \frac{1}{2}\int_{\mathbb{R}^d} |\nabla u_{n_k}|^2 \, \mathrm{d}x \geqslant 0, \end{aligned}$$

  which is in contradiction with $J_M < 0$.

- Any subsequence is not dichotomous.

  Indeed, suppose by contraction that there exist two sequences v_{n_k}, w_{n_k} with disadjoint supports such that

  $$\begin{aligned} &\int_{\mathbb{R}^d} |v_{n_k}|^2 \, \mathrm{d}x \to \alpha M, \quad \int_{\mathbb{R}^d} |w_{n_k}|^2 \, \mathrm{d}x \to (1-\alpha)M, \quad \alpha \in (0,1), \\ &\|u_{n_k}\|^{p+1}_{L^{p+1}} - \|v_{n_k}\|^{p+1}_{L^{p+1}} - \|w_{n_k}\|^{p+1}_{L^{p+1}} \to 0, \\ &\liminf_{k\to\infty} [\|\nabla u_{n_k}\|^2_{L^2} - \|\nabla v_{n_k}\|^2_{L^2} - \|\nabla w_{n_k}\|^2_{L^2}] \geqslant 0, \end{aligned}$$

  then

  $$J_M = \lim_{k\to\infty} E(u_{n_k}) \geqslant \limsup_{k\to\infty} [E(v_{n_k}) + E(w_{n_k})] \geqslant J_{\alpha M} + J_{(1-\alpha)M}.$$

  By the homogeneity property of J_M and $J_1 < 0$ we have

  $$1 \leqslant \alpha^{\frac{1-s_c}{|s_c|}} + (1-\alpha)^{\frac{1-s_c}{|s_c|}}, \text{ with } \frac{1-s_c}{|s_c|} > 1,$$

  which is an contradiction with the fact that if $\theta > 1$, then $f(\alpha) := \alpha^\theta + (1-\alpha)^\theta < f(0) = f(1) = 1$ for all $\alpha \in (0,1)$.

Hence by Lemma 4.3, there exist $(x_k) \subset \mathbb{R}^d$ such that $u_{n_k}(x - x_k) \to u$ in $L^2(\mathbb{R}^d) \cap L^{p+1}(\mathbb{R}^d)$, and hence

$$J_M = \lim_{k\to\infty} E(u_{n_k}) = \lim_{k\to\infty} E(u_{n_k}(x - x_k)) \geqslant E(u).$$

Therefore $u \in H^1(\mathbb{R}^d)$ is a minimiser and $\lim_{k\to\infty} E(u_{n_k}) = E(u)$, $u_{n_k}(x - x_k) \to u$ in $H^1(\mathbb{R}^d)$.

Step 3 Classification of the minimisers

We follow the strategy in Subsection 4.1 to classify the minimisers of the minimisation problem (4.15):

- If u is a minimiser of (4.15), then by Lemma 4.2, we know that $|u| \geqslant 0$ is also a minimiser of (4.15).
- If $u \geqslant 0$ is a minimiser of (4.15), then (**Exercise**) we follow the idea in the proof of Proposition 4.3 to derive the existence of $\tilde{\mu} = \tilde{\mu}(M) \in \mathbb{R}$ (independent of the minimisers) such that

  $$\Delta u + u^p = \tilde{\mu} u, \quad u \geqslant 0, \quad u \in H^1(\mathbb{R}^d). \tag{4.17}$$

  Hence (**Exercise**) we derive (with the notations a, b given in (4.16))

  $$\begin{aligned} \frac{1}{2}\int_{\mathbb{R}^d} |\nabla u|^2 \,\mathrm{d}x \;&= \frac{a}{p+1}\int_{\mathbb{R}^d} u^{p+1} \,\mathrm{d}x\,, \\ \tilde{\mu}\frac{1}{2}\int_{\mathbb{R}^d} |u|^2 \;&= \frac{b}{p+1}\int_{\mathbb{R}^d} u^{p+1} \,\mathrm{d}x\,, \end{aligned}$$

  by testing (4.17) by $u \geqslant 0$ and by $(\frac{d}{2} + x \cdot \nabla)u$ respectively. Therefore

  $$J_M = E(u) = \frac{a-1}{p+1}\int_{\mathbb{R}^d} u^{p+1} \,\mathrm{d}x \; < 0 \text{ for } 1 < p < 1 + \frac{4}{d} \text{ s.t. } a < 1,$$

  and thus

  $$\tilde{\mu} = \tilde{\mu}(M) = \frac{2bJ_M}{(a-1)M} = \frac{2b}{a-1} J_1 M^{-\frac{1}{s_c}} > 0.$$

- If $u \in H^1$, $u \geqslant 0$ satisfies (4.17), then the rescaled solution $\tilde{u}_{\mu^{-1}}(x) = \frac{1}{\mu^{\frac{2}{p-1}}} u(\frac{x}{\mu})$, $\mu = \sqrt{\tilde{\mu}}$ satisfies the equation (4.13)

  $$\Delta v + v^p = v, \quad v \geqslant 0, \quad v \in H^1(\mathbb{R}^d). \tag{4.18}$$

  We have the nontrivial symmetry result established by [Gidas, Ni and Nirenberg, 1979] which we do not prove here:

If v satisfies (4.18), then there exists $x_0 \in \mathbb{R}^d$ such that $v(x - x_0) \in H^1_r(\mathbb{R}^d)$.
Hence by Propositions 4.4 and 4.5, the solution of (4.18) is indeed unique:

$$v(x - x_0) = \tilde{v}(|x - x_0|) = Q(|x - x_0|), \text{ for some } x_0 \in \mathbb{R}^d,$$

where Q is the fundamental solution in Proposition 4.5.

– Conclusion: If u is a minimiser of the minimiser problem (4.15), then there exist $(\gamma_0, x_0) \in \mathbb{R} \times \mathbb{R}^d$ such that

$$u(x) = Q_\mu(x - x_0)e^{i\gamma_0}, \text{ with } Q_\mu(x) = \mu^{\frac{2}{p-1}}Q(\mu x),$$

where

$$\mu = \mu(M) = \frac{\mu(M)}{\mu(\|Q\|^2_{L^2})} = (\frac{\tilde{\mu}(M)}{\tilde{\mu}(\|Q\|^2_{L^2})})^{\frac{1}{2}} = (\frac{M}{\|Q\|^2_{L^2}})^{-\frac{1}{2s_c}}.$$

□

4.3.2 Orbital stability

Recall the *"natural"* stability notion of the solitary waves $e^{it}Q(r)$ in H^1 for $1 < p < 1 + \frac{4}{d}$:
Let $u_0 \in H^1$ and let $u \in C(\mathbb{R}; H^1)$ be the global solution of (NLS) with the initial data u_0. Then for all $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that for all $u_0 \in H^1$:

$$\|u_0 - Q\|_{H^1} < \delta \text{ implies } \sup_{t \geqslant 0}\|u(t, x) - e^{it}Q(x)\|_{H^1} < \varepsilon.$$

This *strong* stability property is *not* suitable for (NLS) by virtue of the symmetries of the equation (NLS). Indeed the scaling invariance and the Galilean invariance in Subsection 1.1.3 supply two obvious examples of strong instability (**Exercise**):

- By scaling symmetry, for any $\lambda > 0$, there exists a solution $u_\lambda(t, x) = \lambda^{\frac{2}{p-1}}Q(\lambda x)e^{i\lambda^2 t}$ of (NLS) with the initial data $(u_0)_\lambda(x) = \lambda^{\frac{2}{p-1}}Q(\lambda x)$. We have

$$\|(u_0)_\lambda - Q\|_{H^1} \to 0 \text{ as } \lambda \to 1,$$

while for any $\lambda \neq 1$,

$$\sup_t\|u_\lambda(t, x) - e^{it}Q(x)\|_{H^1} \geqslant \|Q\|_{H^1}.$$

- By Galilean invariance, for any $v \in \mathbb{R}^d$, there exists a solution $u_v = e^{i(x\cdot v - |v|^2 t + t)}Q(x - 2vt)$ of (NLS) with the initial data $(u_0)_v = e^{iv\cdot x}Q(x)$. We have

$$\|(u_0)_v - Q\|_{H^1} \to 0 \text{ as } |v| \to 0,$$

while whenever $v \neq 0$,

$$\sup{}_t \|u_v(t,x) - e^{it}Q(x)\|_{H^1} \geqslant \|Q\|_{H^1}.$$

By the variational characterisation of the solitary waves in Theorem 4.2, we have the orbital stability results:

Theorem 4.3 (Orbital stability of the solitary waves)**.** *Let* $1 < p < 1 + \frac{4}{d}$, $\kappa = -1$. *Then for all* $\varepsilon > 0$, *there exists* $\delta(\varepsilon) > 0$ *such that for all* $\|u_0 - Q\|_{H^1(\mathbb{R}^d)} < \delta$, *there exist* $(x(t), \gamma(t)) \in \mathbb{R}^d \times \mathbb{R}$ *so that the corresponding solution* $u \in C(\mathbb{R}; H^1)$ *of* (NLS) *satisfying*

$$\sup{}_t \|u(t,x) - Q(x - x(t))e^{i\gamma(t)}\|_{H^1(\mathbb{R}^d)} < \varepsilon.$$

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Proof. Suppose by contradiction that there exist $\varepsilon_0 > 0$, a sequence of times $t_n \geqslant 0$ and a sequence of solutions $u_n \in C(\mathbb{R}; H^1)$ such that

$$\|u_n(0,x) - Q\|_{H^1} \to 0 \text{ as } n \to \infty,$$
$$\|u_n(t_n,x) - Q(x - x_0)e^{i\gamma_0}\|_{H^1} > \varepsilon_0 > 0, \quad \forall x_0 \in \mathbb{R}^d, \forall \gamma_0 \in \mathbb{R}.$$

Then we have

$$\|u_n(0,\cdot)\|_{L^2}^2 \to \|Q\|_{L^2}^2 := M, \quad E(u_n(0,x)) \to E(Q) = J_M.$$

By the mass and energy conservation laws, we have

$$\|u_n(t_n,\cdot)\|_{L^2}^2 = \|u_n(0,\cdot)\|_{L^2}^2 \to M, \quad E(u_n(t_n,x)) = E(u_n(0,x)) \to J_M,$$

and hence by Theorem 4.2, there exist $(\phi(n)) \subset \mathbb{N}$, $(x_{\phi(n)}) \subset \mathbb{R}^d$, $(\gamma_{\phi(n)}) \subset \mathbb{R}$ such that

$$u_n(t_n, x - x_{\phi(n)})e^{i\gamma_{\phi(n)}} \to Q \text{ in } H^1(\mathbb{R}^d), \text{ as } n \to \infty,$$

which is in contradiction to the assumption. □

Remark 4.4. *If $p \geqslant 1 + \frac{4}{d}$, $\kappa = -1$ then the solitary wave $u(t,x) = e^{it}Q(x)$ is unstable in the sense that there exists $(Q_n)_n \subset H^1(\mathbb{R}^d)$ such that $Q_n \to Q$ in $H^1(\mathbb{R}^d)$ while the corresponding solution $u_n(t,x)$ blows up in finite time. Indeed, as $\Delta Q - Q + Q^p = 0$, if $p = 1 + \frac{4}{d}$, then*

$$E(Q) = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla Q|^2 \,\mathrm{d}x - \frac{1}{p+1}\int_{\mathbb{R}^d} |Q|^{p+1} \,\mathrm{d}x = 0$$

and $E(\lambda Q) < 0$ for any $\lambda > 1$, and hence we have the blowup results by Theorem 3.1 for the solutions with initial data $(1 + \frac{1}{n})Q$. Similarly we notice that if $p > 1 + \frac{4}{d}$, then with $a = \frac{d(p-1)}{4} > 1$

$$\frac{1}{2}\int_{\mathbb{R}^d} |\nabla Q|^2 \,\mathrm{d}x - \frac{a}{p+1}\int_{\mathbb{R}^d} |Q|^{p+1} \,\mathrm{d}x = 0.$$

For any $\varepsilon > 0$ we can take λ_ε small enough such that $E(Q_{\lambda_\varepsilon}) = \lambda_\varepsilon^{2(\frac{2}{p-1} - \frac{d}{2} + 1)} E(Q) < \varepsilon$ where $Q_\lambda = \lambda^{\frac{2}{p-1}} Q(\lambda x)$ and hence there exists λ larger but close to 1 such that $E(\lambda Q_{\lambda_\varepsilon}) < 0$.

4.4 Linearized operators

We linearize the (NLS) equation around the solitary wave solution $e^{it}Q(x)$, that is, we consider the solutions of the form

$$u(t,x) = e^{it}(Q(x) + h(t,x)),$$

and search for the equation satisfied by h:

$$\partial_t h = \text{linear term of } h + \text{ nonlinear terms.}$$

More precisely, the linear part of the equation for h reads as

$$\mathcal{L}h = -i\big[(-\Delta + 1 - Q^{p-1})h - \frac{p-1}{2}Q^{p-1}(h + \bar{h})\big],$$

which can be rewritten into a matrix operator acting on $\begin{pmatrix} \operatorname{Re} h \\ \operatorname{Im} h \end{pmatrix}$:

$$\mathcal{L} = \begin{bmatrix} 0 & L_- \\ -L_+ & 0 \end{bmatrix}, \quad L_+ = -\Delta + 1 - pQ^{p-1}, \quad L_- = -\Delta + 1 - Q^{p-1}. \tag{4.19}$$

We are going to study the spectral properties of $\mathcal{L}$ and the two operators L_+, L_-, which are self-adjoint Schrödinger operators with continuous spectrum $[1, \infty)$ and with finitely many eigenvalues below 1. These spectral properties play a central role in the stability theory.

4.4.1 Nullspaces

It is straightforward to compute that

$$L_-Q = 0, \quad L_+(\Lambda Q) = -2Q, \text{ with } \Lambda = \frac{2}{p-1} + x \cdot \nabla. \tag{4.20}$$

Indeed, $L_-Q = 0$ is the equation for Q. For the second equation, since

$$\Lambda Q = -\frac{d}{d\lambda}|_{\lambda=1} Q_\lambda, \quad Q_\lambda(x) = \lambda^{-\frac{2}{p-1}} Q(\lambda^{-1}x),$$

we apply $\frac{d}{d\lambda}|_{\lambda=1}$ to the equation $\Delta Q_\lambda + Q_\lambda^p = \lambda^{-2} Q_\lambda$ to arrive at $L_+(\Lambda Q) = -2Q$.

We take the derivative ∇ and the multiplication x to the Q equation: $(-\Delta + 1 - Q^{p-1})Q = 0$ to derive

$$L_+ \nabla Q = 0, \quad L_-(xQ) = -2\nabla Q. \tag{4.21}$$

Lemma 4.4. *If* $1 < p < 2^* - 1$, *then*

$$\inf_{(f,Q^p)=0} (L_+ f, f) = 0.$$

If $1 < p \leqslant 1 + \frac{4}{d}$, *then*

$$\inf_{(f,Q)=0} (L_+ f, f) = 0.$$

Proof. We have the following result from [Weinstein 1983 CMP] (we do not give a proof here): For $1 < p < 2^* - 1$,

$$J[u] = \frac{\|\nabla u\|_{L^2}^{2a} \|u\|_{L^2}^{2b}}{\|u\|_{L^{p+1}}^{p+1}}, \quad a = \frac{d(p-1)}{4}, \quad b = \frac{d+2-(d-2)p}{4}, \quad u \in H^1(\mathbb{R}^d)$$

attains its minimum $\alpha = \frac{\|\psi\|_{L^2}^{p-1}}{(p+1)/2}$ at $\frac{\psi}{\|\psi\|_{L^2}}$ where ψ satisfies

$$a\Delta\psi - b\psi + \psi^p = 0, \quad \psi \geqslant 0, \quad \psi \in H_r^1(\mathbb{R}^d).$$

Recall the fundamental solution Q which satisfies $\Delta Q - Q + Q^p = 0$, then

$$\psi = b^{\frac{1}{p-1}} Q(\sqrt{\frac{b}{a}} x).$$

The minimisation inequality

$$\frac{d^2}{d\varepsilon^2}|_{\varepsilon=0} J(\psi + \varepsilon\eta) \geqslant 0, \quad \forall \eta \in C_0^\infty(\mathbb{R}^d)$$

reads as (**Exercise**)

$$c(L_+\eta,\eta) \geqslant \frac{1}{a}\Big(\int \eta\Delta Q\Big)^2 + \frac{1}{b}\Big(\int \eta Q\Big)^2 - \Big(\int \eta Q^p\Big)^2,$$

where $c = \frac{1}{p+1}\int Q^{p+1} = \frac{1}{2a}\int |\nabla Q|^2 = \frac{1}{2b}\int |Q|^2 > 0$.

If $1 < p < 2^* - 1$, then

$$(L_+\eta,\eta) \geqslant 0, \quad \text{if } (\eta, Q^p) = 0.$$

If $1 < p \leqslant 1 + \frac{4}{d}$, then $a \leqslant 1$, such that by use of $\Delta Q = -Q^p + Q$,

$$(L_+\eta,\eta) \geqslant 0, \quad \text{if } (\eta, Q) = 0.$$

On the other hand, by $L_+\nabla Q = 0$, we have $(L_+\nabla Q, \nabla Q) = 0$ and $(\nabla Q, Q^p) = (\nabla Q, Q) = 0$. □

We have immediately $L_+|_{\{Q\}^\perp} \geqslant 0$. We also have the following information on the nullspaces of $L_\pm$, which can be found in [Weinstein 1985 SIAM] or [Chang-Gustafson-Nakanishi-Tsai 2007 SIAM].

Lemma 4.5. *Let $1 < p < 2^* - 1$. Then*

$$\begin{aligned}
&L_- \geqslant 0, \quad N(L_-) = \operatorname{span}\{Q\}, \quad L_-|_{\{Q\}^\perp} > 0,\\
&N(L_+) = \operatorname{span}\{\nabla Q\}, \quad (Q, L_+Q) < 0, \quad L_+|_{\{Q^p\}^\perp} \geqslant 0\\
&L_+|_{\{Q\}^\perp} \geqslant 0 \text{ if } 1 < p \leqslant 1 + \frac{4}{d}.
\end{aligned}$$

Let $N_g(A) = \cup_{k=1}^{\infty} N(A^k)$ denote the generalized nullspace of A. Then for $p \neq 1 + \frac{4}{d}$,

$$N_g(\mathcal{L}) = \operatorname{span}\left\{\begin{pmatrix}0\\Q\end{pmatrix}, \quad \begin{pmatrix}0\\xQ\end{pmatrix}, \quad \begin{pmatrix}\nabla Q\\0\end{pmatrix}, \quad \begin{pmatrix}\Lambda Q\\0\end{pmatrix}\right\}.$$

Let us define the functional space

$$\mathcal{H} := (H^1 \times H^1) \cap [N_g(\mathcal{L}^*)]^\perp,$$

then its elements satisfy the following orthogonality relations:

Corollary 4.1. *Let $\begin{pmatrix}f\\g\end{pmatrix} \in \mathcal{H}$ and $1 < p < 2^* - 1$, $p \neq 1 + \frac{4}{d}$, then the following $2d+2$ orthogonality relations hold:*

$$\begin{aligned}
&(f, Q) = 0, \quad (f, xQ) = 0,\\
&(g, \nabla Q) = 0, \quad (g, \Lambda Q) = 0.
\end{aligned}$$

Remark 4.5. *If $p = 1 + \frac{4}{d}$ is the mass critical exponent, then*

$$L_-(|x|^2 Q) = -4\Lambda Q, \quad L_+\rho = |x|^2 Q,$$

for some radial function $\rho(x)$ (for which we do not know an explicit formula in terms of Q). The generalized nullspace of $\mathcal{L}$ reads as

$$N_g(\mathcal{L}) = \operatorname{span}\left\{ \begin{pmatrix} 0 \\ Q \end{pmatrix}, \quad \begin{pmatrix} 0 \\ xQ \end{pmatrix}, \quad \begin{pmatrix} 0 \\ |x|^2 Q \end{pmatrix}, \quad \begin{pmatrix} \nabla Q \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \Lambda Q \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \rho \\ 0 \end{pmatrix} \right\}.$$

If $\begin{pmatrix} f \\ g \end{pmatrix} \in \mathcal{H} := (H^1 \times H^1) \cap [N_g(\mathcal{L}^*)]^\perp$, the above $2d+2$ orthogonality relations in Proposition 4.1 together with the following two orthogonality relations hold:

$$(f, |x|^2 Q) = 0, \quad (g, \rho) = 0.$$

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4.4.2 Coercivity

Theorem 4.4. *Let $1 < p < 1 + \frac{4}{d}$. Then there exist positive constants K, K' such that*

$$K(\|f\|_{H^1}^2 + \|g\|_{H^1}^2) \leqslant (L_+ f, f) + (L_- g, g) \leqslant K'(\|f\|_{H^1}^2 + \|g\|_{H^1}^2), \quad \forall \begin{pmatrix} f \\ g \end{pmatrix} \in \mathcal{H}.$$

Proof. It is obvious to see from

$$(L_+ f, f) = \int |\nabla f|^2 + |f|^2 - pQ^{p-1}|f|^2, \quad (L_- g, g) = \int |\nabla g|^2 + |g|^2 - Q^{p-1}|g|^2$$

that the upper bound holds true.

We claim that

$$\tau := \inf_{\mathcal{A}_+}(L_+ f, f) > 0,$$
$$\mathcal{A}_+ = \{f \in H^1 \mid \|f\|_{L^2} = 1, (f, Q) = 0, (f, xQ) = 0\}.$$

Then $L_+|_{\{Q, xQ\}^\perp} > 0$ and

$$(L_+ f, f) \geqslant \tau \|f\|_{L^2}^2, \quad \text{if } (f, Q) = (f, x_j Q) = 0, \forall 1 \leqslant j \leqslant d,$$

such that

$$p\int Q^{p-1}|f|^2 \leqslant C\|f\|_{L^2}^2 \leqslant \tau^{-1}C(L_+f,f)$$
$$\Longrightarrow (\tau^{-1}C+1)(L_+f,f) \geqslant \int |\nabla f|^2+|f|^2.$$

That is, there exists $K>0$ such that $(L_+f,f)\geqslant K\|f\|_{H^1}^2$.

Indeed, by Lemma 4.5: $L_+|_{\{Q\}^\perp}\geqslant 0$, it suffices to show $\tau\neq 0$. We assume that $\tau=0$, and there exists a minimizing sequence $\{f_n\}$ with $\|f_n\|_{L^2}=1$, $(f_n,Q)=(f_n,xQ)=0$, such that

$$(L_+f_n,f_n)=\int |\nabla f_n|^2+|f_n|^2-pQ^{p-1}|f_n|^2\to 0.$$

Then for any $\eta>0$, we can assume that all f_n satisfy

$$0<1\leqslant \int |\nabla f_n|^2+|f_n|^2 \leqslant p\int Q^{p-1}|f_n|^2+\eta.$$

The righthand side is uniformly bounded by $\|f_n\|_{L^2}=1$, and hence $\{f_n\}$ converges weakly to $f_*\in H^1$, which satisfies the orthogonality relations. By the decay property of Q, we also have

$$\int Q^{p-1}|f_n|^2\to\int Q^{p-1}|f_*|^2,$$

and hence $f_*\neq 0$ since η is arbitrary.

Without loss of generality we can assume that f_* is admissible: $\|f_*\|_{L^2}=1$ and the minimum is attained at f_*. Indeed, by Fatou's Lemma, $\|f_*\|_{L^2}\leqslant 1$. If $\|f_*\|_{L^2}<1$, then we simply take $\frac{f_*}{\|f_*\|_{L^2}}$, which is admissible. Since

$$(L_+f_*,f_*)\leqslant \liminf_{n\to\infty}\int |\nabla f_n|^2+|f_n|^2-pQ^{p-1}|f_n|^2=0,$$

we also have $(L_+f_*,f_*)=(L_+\frac{f_*}{\|f_*\|_{L^2}},\frac{f_*}{\|f_*\|_{L^2}})=0=\tau$.

Since the minimum is attained at an admissible function $f_*\neq 0$, there exists $(f_*,\lambda,\beta,\gamma)$ among the critical points of the Lagrange multiplier problem (**Exercise**)

$$(L_+-\lambda)f=\beta Q+\gamma\cdot xQ,\quad \|f\|_{L^2}=1,\quad (f,Q)=(f,x_jQ)=0.$$

Hence $\lambda=(L_+f_*,f_*)=\tau=0$ is a critical value and

$$0=(f_*,L_+\nabla Q)=(L_+f_*,\nabla Q)=\beta(Q,\nabla Q)+(\gamma\cdot xQ,\nabla Q),$$

that is, $0 = (\gamma \cdot xQ, \nabla Q)$ and thus $\gamma = 0$, $L_+ f_* = \beta Q$. Since

$$L_+ \Lambda Q = -2Q, \quad N(L_+) = \operatorname{span}\{\nabla Q\},$$

we have

$$f_* = -\frac{\beta}{2}\Lambda Q + \theta \cdot \nabla Q, \quad \theta \in \mathbb{R}^d .$$

Since $(f_*, xQ) = 0$, $\theta = 0$ and hence $(f_*, Q) = -\frac{\beta}{2}(\frac{2}{p-1} - \frac{d}{2})\|Q\|_{L^2}^2$. If $1 < p < 1 + \frac{4}{d}$, then $f_* = 0$ due to $(f_*, Q) = 0$. This is a contradiction. Therefore $\tau > 0$.

Now we turn to the operator L_-. We claim that

$$\begin{aligned} &\mu := \inf_{\mathcal{A}_-}(L_- g, g) > 0, \\ &\mathcal{A}_- = \{g \in H^1 \mid \|g\|_{L^2} = 1,\ (g, \nabla Q) = 0,\ (g, \Lambda Q) = 0\}. \end{aligned}$$

Indeed if $\mu = 0$, then the argument as above show that there exists $g_* \in \mathcal{A}_-$ such that $(L_- g_*, g_*) = \mu = 0$. Since $L_- \geqslant 0$ and $N(L_-) = \operatorname{span}\{Q\}$, $g_* = \frac{Q}{\|Q\|_{L^2}}$, which implies $(g_*, \Lambda Q) = (\frac{2}{p-1} - \frac{d}{2})\|Q\|_{L^2}^2$. This is a contradiction with the orthogonality relation if $1 < p < 1 + \frac{4}{d}$. Hence $\mu > 0$, such that $(L_- g, g) \geqslant \mu \|g\|_{L^2}^2$ for g satisfying the orthogonality relations and thus $(L_- g, g) \geqslant K\|g\|_{H^1}^2$ for some $K > 0$.

□

Remark 4.6. *The estimates can be easily extended to the critical case $p = 1 + \frac{4}{d}$, with additional orthogonality relations $(f, |x|^2 Q) = 0$ and $(g, \rho) = 0$ in the admissible sets $\mathcal{A}_+$, $\mathcal{A}_-$ respectively. By the same argument as above, if $\tau = 0$, then the minimum is attained at an admissible function f_*: $(L_+ f_*, f_*) = 0$, which is the critical point of the Lagrange multiplier problem*

$$\begin{aligned} &(L_+ - \lambda) f = \beta Q + \gamma \cdot xQ + \delta |x|^2 Q, \\ &\|f\|_{L^2} = 1, \quad (f, Q) = (f, x_j Q) = (f, |x|^2 Q) = 0. \end{aligned}$$

Hence since $\tau = 0$, $\lambda = 0$ and since $(L_+ f, \nabla Q) = 0$, $\gamma = 0$. Since $L_+ \Lambda Q = -2Q$, $L_+ \rho = -|x|^2 Q$, $N(L_+) = \operatorname{span}\{\nabla Q\}$,

$$f_* = -\frac{\beta}{2}\Lambda Q - \delta \rho + \theta \cdot \nabla Q.$$

Since $(\Lambda Q, |x|^2 Q) = -\int |x|^2 Q^2 \neq 0$, $(\rho, Q) \neq 0$, $(\nabla Q, xQ) \neq 0$, the orthogonality relations imply $f_* = 0$. This is a contradiction and hence $\tau > 0$. If $\mu = 0$, then $(g_*, \rho) = -\frac{1}{\|Q\|_{L^2}} \int |x|^2 Q^2 \neq 0$. This is also a contradiction with the orthogonality relations.

Corollary 4.2. *Let $1 < p \leqslant 1 + \frac{4}{d}$ and $h_0 \in \mathcal{H}$, $r(t) \in \mathcal{H}$. Then the solution h to the linear system*

$$\partial_t h = \mathcal{L}h + r, \quad h|_{t=0} = h_0,$$

satisfies $h = f + ig \in \mathcal{H}$ and

$$K\|h(t)\|_{H^1}^2 \leqslant (L_+ f, f) + (L_- g, g) \leqslant K'\|h(t)\|_{H^1}^2.$$

In particular, if $r(t) = 0$,

$$K\|h(t)\|_{H^1}^2 \leqslant (L_+ f, f) + (L_- g, g) = (L_+ f_0, f_0) + (L_- g_0, g_0) \leqslant K'\|h_0\|_{H^1}^2.$$

Proof. Check that the linear flow maps from $\mathcal{H}$ to $\mathcal{H}$ and hence the estimate holds for any time t. Check that the quantity $(L_+ f, f) + (L_- g, g)$ is conserved by the homogeneous flow. (**Exercise**) □

4.4.3 Modulational stability

Let $\lambda_0, \gamma_0, \xi_0 = (\xi_0^1, \cdots, \xi_0^d), \theta_0 = (\theta_0^1, \cdots, \theta_0^d)$ be real numbers, then (**Exercise**) the functions

$$\lambda_0^{\frac{2}{p-1}} Q(\lambda_0(\theta(x,t) - \theta_0)) e^{i(\xi_0 \cdot (\theta(x,t) - \theta_0) + (\gamma(x,t) - \gamma_0))}$$

form a $(2d+2)$-parameter family of solutions of (NLS), $1 < p < 2^* - 1$ if the following relations hold for $\theta = (\theta^1, \cdots, \theta^d)$ and γ:

$$\frac{\partial \theta^i}{\partial t} = -2\xi_0^i, \quad \frac{\partial \theta^i}{\partial x^j} = \delta_i^j, \quad \frac{\partial \gamma}{\partial t} = \lambda_0^2 + |\xi_0|^2, \quad \frac{\partial \gamma}{\partial x^j} = 0. \tag{4.22}$$

Let us consider the perturbed Cauchy problem of the (NLS):

$$i\partial_t u^\varepsilon + \Delta u^\varepsilon = -|u^\varepsilon|^{p-1} u^\varepsilon, \quad u^\varepsilon|_{t=0} = Q(x) + \varepsilon h_0(x), \quad p < 1 + \frac{4}{d}.$$

Let the $2d + 2$ parameters vary slowly:

$$\lambda_0 = \lambda_0(T), \quad \gamma_0 = \gamma_0(T), \quad \xi_0 = \xi_0(T), \quad \theta_0 = \theta_0(T), \text{ where } T = \varepsilon t,$$

and we are going to determine the slow time evolution of these parameters such that we can expand u^ε as

$$u^\varepsilon(t,x) = \big(\lambda_0^{\frac{2}{p-1}} Q(\lambda_0(\theta - \theta_0)) + \varepsilon h_1\big) e^{i(\xi_0 \cdot (\theta - \theta_0) + (\gamma - \gamma_0))},$$
$$h_1 = h_1(\tau, \Theta), \quad \tau = \int_0^t \lambda_0^2 \, \mathrm{d}t', \quad \Theta = \lambda_0(\theta - \theta_0),$$

where θ, γ satisfy (4.22) and $h_1 \in \mathcal{H}$ for $t > 0$ and for any $T_0 > 0$,

$$\sup_{0\leqslant t\leqslant T_0/\varepsilon} \|\varepsilon h_1(t)\|_{H^1} = \alpha(\varepsilon), \quad \alpha(\varepsilon) \to 0 \text{ as } \varepsilon \to 0. \tag{4.23}$$

Indeed, we substitute the above expansion of u^ε with $h_1 = h + h_0$ into the perturbed Cauchy problem and balance the terms of order ε to derive (**Exercise**)

$$\partial_\tau h = \mathcal{L}h + r, \quad h|_{t=0} = 0, \quad h = (f,g)^t,$$

where

$$r = (r_1, r_2)^t + \mathcal{L}h_0, \quad r_1 = -\lambda_0^{\frac{2}{p-1}-3}\dot{\lambda}_0(\frac{2}{p-1}Q + \Theta \cdot \nabla Q) + \lambda_0^{\frac{2}{p-1}-1}\dot{\theta}_0 \cdot \nabla Q,$$

$$r_2 = -\lambda_0^{\frac{2}{p-1}-3}\dot{\xi}_0 \cdot \Theta Q + (\xi_0 \cdot \dot{\theta}_0 + \dot{\gamma}_0)\lambda_0^{\frac{2}{p-1}-2}Q.$$

If $r(t) \in \mathcal{H}$, then (r_1, r_2) have to satisfy

$$(r_1, Q) = -(L_-\mathrm{Im}\, h_0, Q), \text{ i.e. } -\lambda_0^{\frac{2}{p-1}-3}(\frac{2}{p-1} - \frac{d}{2})\dot{\lambda}_0\|Q\|_{L^2}^2 = 0,$$

$$(r_1, xQ) = -(L_-\mathrm{Im}\, h_0, xQ), \text{ i.e. } -\frac{1}{2}\lambda_0^{\frac{2}{p-1}-1}\dot{\theta}_0\|Q\|_{L^2}^2 = 2(\mathrm{Im}\, h_0, \nabla Q),$$

$$(r_2, \nabla Q) = (L_+\mathrm{Re}\, h_0, \nabla Q), \text{ i.e. } -\lambda_0^{\frac{2}{p-1}+3}\dot{\xi}_0\|Q\|_{L^2}^2 = 0,$$

$$(r_2, \Lambda Q) = (L_+\mathrm{Re}\, h_0, \Lambda Q), \text{ i.e. } \lambda_0^{\frac{2}{p-1}-2}(\frac{2}{p-1} - \frac{d}{2})(\xi_0 \cdot \dot{\theta}_0 + \dot{\gamma}_0)\|Q\|_{L^2}^2 = -2(\mathrm{Re}\, h_0, Q),$$

that is,

$$\dot{\lambda}_0 = 0 \Rightarrow \lambda_0 = 1, \quad \dot{\xi}_0 \Rightarrow \xi_0 = 0,$$

$$\text{and } \theta_0 = -4\varepsilon\|Q\|_{L^2}^{-2}(\mathrm{Im}\, h_0, \nabla Q)t, \quad \gamma_0 = -2\varepsilon(\frac{2}{p-1} - \frac{d}{2})^{-1}\|Q\|_{L^2}^{-2}(\mathrm{Re}\, h_0, Q)t. \tag{4.24}$$

If the parameters θ_0, γ_0 grow slowly in time as in (4.24) such that $r \in \mathcal{H}$, then by Corollary 4.2, $h = (f,g)^t \in \mathcal{H}$ and

$$K\|h(\tau)\|_{H^1}^2 \leqslant (L_+f, f) + (L_-g, g) := A(h).$$

Since

$$h(\tau) = e^{\mathcal{L}\tau}\int_0^\tau e^{-\mathcal{L}s}r(\varepsilon s)ds = (f,g)^t,$$

and $e^{\mathcal{L}\tau}$ is a unitary group acting on $\mathcal{H}$, we have

$$K\|\varepsilon h(\tau)\|_{H^1}^2 \leqslant \varepsilon^2 A(h) \leqslant A\big(\varepsilon\int_0^\tau e^{-\mathcal{L}s}r(\varepsilon s)ds\big).$$

The following is a rough idea without proof: Heuristically, $\varepsilon \int_0^\tau e^{-\mathcal{L}s} r(\varepsilon s) ds$ tends to the projection of r onto the nullspace of $\mathcal{L}$ (as r is "almost constant" and by the mean ergodic theorem) and hence tends to 0 as $N(\mathcal{L}) \cap \mathcal{H} = \{0\}$. Therefore the modulational stability result (4.23) holds, under modulational ordinary differential equations in time for the $d+1$ parameters γ_0, θ_0 alone. This is consistent with the nonlinear stability result. In particular, if initially $u^\varepsilon|_{t=0} = Q(x) + \varepsilon h_0$, then

$$u^\varepsilon(t,x) = e^{i(t-\gamma_0(T))}(Q(x - x_0(T)) + \varepsilon h_1(t, x - x_0(T))), \quad T = \varepsilon t, \text{ where}$$

$$x_0(T) = -4\|Q\|_{L^2}^{-2}(\operatorname{Im} h_0, \nabla Q)T, \quad \gamma_0(T) = -2(\frac{2}{p-1} - \frac{d}{2})^{-1}\|Q\|_{L^2}^{-2}(\operatorname{Re} h_0, Q)T,$$

such that (4.23) holds.

We can also consider the perturbation added in the NLS equation to derive (from the $2d+2$ orthogonality conditions) the modulational ordinary differential equations in time for the $2d+2$ parameters $\lambda_0, \gamma_0, \xi_0, \theta_0$. These modulational stability results can be found in [Weinstein 1985 SIAM J. Math. Anal.].

4.5 Lyapunov method

Let $1 < p < 1 + \frac{4}{d}$ and we introduce a Lyapunov functional

$$H(u) = 2E(u) + M(u) = \int_{\mathbb{R}^d} |\nabla u|^2 - \frac{2}{p+1}\int_{\mathbb{R}^d} |u|^{p+1} + \int_{\mathbb{R}^d} |u|^2,$$

which is invariant under translation and phase rotation.

For any initial data $u_0 \in H^1$, by Theorem 2.9 there exists a global solution $u \in C(\mathbb{R}; H^1)$ of the Cauchy problem of (NLS). We write

$$u(t, x + x(t))e^{i\gamma(t)} = e^{it}(Q(x) + h(t,x)),$$

then by the mass and energy conservation laws,

$$H(u_0) - H(Q) = H(u(t, \cdot + x(t))e^{i\gamma(t)}) - H(e^{it}Q) = H(e^{it}(Q+h)) - H(e^{it}Q),$$

which, by the elliptic equation $-\Delta Q + Q = Q^p$, implies (**Exercise**)

$$H(u_0) - H(Q) \geqslant (L_+ \operatorname{Re} h, \operatorname{Re} h) + (L_- \operatorname{Im} h, \operatorname{Im} h) - C(\|h\|_{H^1}^{2+\theta} + \|h\|_{H^1}^6),$$

with $\theta > 0$.

We claim that if $x(t), \gamma(t)$ have been chosen to minimize

$$\|u(t, \cdot + x(t))e^{i\gamma(t)} - Q\|_{H^1}^2 = \|h\|_{H^1}^2,$$

with the restriction $\|u\|_{L^2} = \|Q\|_{L^2}$, then for $h = f + ig$, $f, g \in \mathbb{R}$

$$(L_+ f, f) + (L_- g, g) \geqslant K\|h\|_{H^1}^2 - C(\|h\|_{H^1}^3 + \|h\|_{H^1}^4). \qquad (4.25)$$

Thus

$$H(u_0) - H(Q) \geqslant K\|h\|_{H^1}^2 - C(\|h\|_{H^1}^{2+\theta} + \|h\|_{H^1}^6) := G(\|h\|_{H^1}^2).$$

Therefore for any small enough $\varepsilon > 0$, by the continuity of H in H^1 there exists $\delta > 0$ such that

$$\text{if } \|h_0\|_{H^1}^2 < \delta \text{ then } H(u_0) - H(Q) < G(\varepsilon) \text{ and hence } \|h\|_{H^1}^2 < \varepsilon,$$

which implies the orbital stability result Theorem 4.3. For general case $\inf_{x_0 \in \mathbb{R}, \gamma_0 \in [0,2\pi)} \|u_0(x + x_0)e^{i\gamma_0} - Q(x)\|_{H^1}^2 < \delta$ without the restriction $\|u_0\|_{L^2} = \|Q\|_{L^2}$, we can take the ground state $\psi_\lambda(t,x) = \frac{1}{\lambda^{\frac{2}{p-1}}} Q(\frac{x}{\lambda}) e^{i\frac{t}{\lambda^2}}$ which has the same L^2-norm as u_0 and $\|\psi_\lambda - Q\|_{H^1} < \frac{1}{2}\varepsilon$. Hence by triangle inequality we still have $\|h\|_{H^1}^2 < \varepsilon$.

We now prove the claim. Minimization of $\|h\|_{H^1}^2$ implies (**Exercise**)

$$\int Q^{p-1} \partial_{x_j} Q f \,\mathrm{d}x = 0, \qquad \int Q^p g \,\mathrm{d}x = 0.$$

Since $L_-|_{\{Q\}^\perp} > 0$, we have the following by $g \in \{Q\}^\perp$

$$(L_- g, g) \geqslant c\|g\|_{L^2}^2 \text{ and hence } (L_- g, g) \geqslant K\|g\|_{H^1}^2.$$

In order to show the positivity estimate for L_+, recall that we restrict ourselves to the case

$$\|u\|_{L^2} = \|Q\|_{L^2}, \text{ such that } (f, Q) = -\frac{1}{2}\big((f,f) + (g,g)\big) = -\frac{1}{2}\|h\|_{L^2}^2.$$

We decompose $f = f_\parallel + f_\perp$ as follows:

$$f_\parallel = (f,Q)Q = -\frac{1}{2}\|h\|_{L^2}^2 Q,$$
$$f_\perp = f - (f,Q)Q = f + \frac{1}{2}\|h\|_{L^2}^2 Q.$$

Hence

$$(L_+ f, f) = (L_+ f_\parallel, f_\parallel) + 2(L_+ f_\parallel, f_\perp) + (L_+ f_\perp, f_\perp),$$

where

$$(L_+ f_\parallel, f_\parallel) = \frac{1}{4}\|h\|_{L^2}^4 (L_+ Q, Q),$$
$$(L_+ f_\parallel, f_\perp) = (f, Q)(L_+ f_\perp, Q) = -\frac{1}{2}\|h\|_{L^2}^2 (L_+ f_\perp, Q)$$
$$= -\frac{1}{2}\|h\|_{L^2}^2 (L_+ f, Q) - \frac{1}{4}\|h\|_{L^2}^4 (L_+ Q, Q) \geqslant -C\|h\|_{L^2}^3,$$
$$(L_+ f_\perp, f_\perp) \geqslant c(f_\perp, f_\perp) = c(f,f) - c(f_\parallel, f_\parallel) = c(f,f) - c\frac{1}{4}\|h\|_{L^2}^4.$$

In the above, the last inequality follows from the proof of Theorem 4.4: the minimum $(L_+ f_\perp, f_\perp) = 0$ can only be attained at $\theta \cdot \nabla Q$ which violates $\int Q^{p-1}\partial_{x_j} Q f = 0$. Therefore

$$(L_+ f, f) \geqslant K\|f\|_{H^1}^2 - C(\|h\|_{L^2}^3 + \|h\|_{L^2}^4),$$

which implies (4.25).

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[02.02.2018]

A Jost solutions

A.1 Time independent case

We look for solutions of the initial value problem for the ODE system (1.15):

$$j_x = \begin{pmatrix} -iz & u \\ \bar{u} & iz \end{pmatrix} j,$$

under the following initial value conditions respectively:

$$j^{-,1}(x; z) = \begin{pmatrix} e^{-izx} \\ 0 \end{pmatrix} + o(1) \text{ as } x \to -\infty,$$

and

$$j^{-,2}(x; z) = \begin{pmatrix} 0 \\ e^{izx} \end{pmatrix} + o(1) \text{ as } x \to -\infty.$$

Indeed, the renormalised solution $l^{-,1}(x; z) = e^{izx} j^{-,1}(x; z)$ satisfies the following integral equation

$$l^{-,1}(x; z) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_{-\infty}^{x} \begin{pmatrix} 0 & u(x_1) \\ e^{2iz(x-x_1)}\bar{u}(x_1) & 0 \end{pmatrix} l^{-,1}(x_1; z)\, \mathrm{d}x_1. \quad \text{(A.26)}$$

By iteration we deduce that

$$l^{-,1}(x;z) = \sum_{n=0}^{\infty} l_n^{-,1}(x;z), \quad l_0^{-,1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$l_{n+1}^{-,1}(x;z) = \int_{-\infty}^{x} \begin{pmatrix} 0 & u(x_1) \\ e^{2iz(x-x_1)}\bar{u}(x_1) & 0 \end{pmatrix} l_n^{-,1}(x_1;z)\, \mathrm{d}x_1,$$

that is,

$$l^{-,1}(x;z) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \int_{-\infty}^{x} e^{2iz(x-x_1)}\bar{u}(x_1)\,\mathrm{d}x_1 \end{pmatrix} + \begin{pmatrix} \int_{-\infty}^{x} u(x_2) \int_{-\infty}^{x_2} e^{2iz(x_2-x_1)}\bar{u}(x_1)\,\mathrm{d}x_1\,\mathrm{d}x_2 \\ 0 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 \\ \int_{-\infty}^{x} e^{2iz(x-x_3)}\bar{u}(x_3) \int_{-\infty}^{x_3} u(x_2) \int_{-\infty}^{x_2} e^{2iz(x_2-x_1)}\bar{u}(x_1)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3 \end{pmatrix}$$

$$+ \begin{pmatrix} \int_{-\infty}^{x} u(x_4) \int_{-\infty}^{x_4} e^{2iz(x_4-x_3)}\bar{u}(x_3) \int_{-\infty}^{x_3} u(x_2) \int_{-\infty}^{x_2} e^{2iz(x_2-x_1)}\bar{u}(x_1)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\,\mathrm{d}x_4 \\ 0 \end{pmatrix} + \cdots \tag{A.27}$$

We just have to show that the above series converges if $u \in L^1(\mathbb{R};\mathbb{C})$: Let

$$m_k = \int_{-\infty}^{x_k} |u(y)|\,\mathrm{d}y, \quad m = \int_{-\infty}^{x} |u(y)|\,\mathrm{d}y,$$

then $\mathrm{d}m_k = |u(x_k)|\,\mathrm{d}x_k$ and $|l^{-,1}(x;z)|$ with $x, z \in \mathbb{R}$ can be controlled by

$$\begin{pmatrix} 1 + \int_0^m \int_0^{m_2} \mathrm{d}m_1\,\mathrm{d}m_2 + \int_0^m \int_0^{m_4} \int_0^{m_3} \int_0^{m_2} \mathrm{d}m_1\,\mathrm{d}m_2\,\mathrm{d}m_3\,\mathrm{d}m_4 + \cdots \\ \int_0^m \mathrm{d}m_1 + \int_0^m \int_0^{m_3} \int_0^{m_2} \mathrm{d}m_1\,\mathrm{d}m_2\,\mathrm{d}m_3 + \cdots \end{pmatrix}$$

and hence by

$$\begin{pmatrix} \sum_{k=0}^{\infty} \frac{m^{2k}}{(2k)!} \\ \sum_{k=0}^{\infty} \frac{m^{2k+1}}{(2k+1)!} \end{pmatrix}$$

which converges since $m \leqslant \|u\|_{L^1}$. The uniqueness result follows similarly by iteration: Let l be the difference between two solutions satisfying the above integral equation (A.26) with the boundary condition $(0,0)^T$, then we substitute this integral equation into itself k times such that $|l(x;z)| \leqslant \|l\|_{L^\infty} \frac{\|u\|_{L^1}^k}{k!}$ which vanishes as $k \to \infty$.

Hence we arrive at a 2×2 fundamental solution matrix:

$$J^-(x;z) = [j^{-,1}(x;z), j^{-,2}(x;z)].$$

with normalization condition

$$\lim_{x\to-\infty} J^-(x;z)e^{izx\sigma_3} = \mathbb{I},\ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Similarly we can define Jost solutions $j^{+,1}(x;z), j^{+,2}(x;z)$ that are normalized in the limit $x \to +\infty$ and the associated normalized fundamental solution matrix

$$J^+(x;z) = [j^{+,1}(x;z), j^{+,2}(x;z)], \text{ with } \lim_{x\to+\infty} J^+(x;z)e^{izx\sigma_3} = \mathbb{I}.$$

There is a scattering matrix $S = S(z)$ such that $J^+(x;z) = J^-(x;z)S(z)$, with $S(z)$ given in (1.24).

A.2 Time dependent case

We are going to show that the matrices

$$W^\pm(t,x;z) = J^\pm(t,x;z)e^{-2iz^2t\sigma_3}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are simultaneous fundamental solutions of the compatible linear systems (1.15)-(1.16) of the Lax pair. Indeed, let $W(t,x;z)$ be the simultaneous fundamental solution matrix of the compatible systems (1.15)-(1.16). Since any solution matrix of (1.15) is a linear combination of $j^{+,1}, j^{+,2}$, there exists time-dependent (and space-independent) matrix $C(t;z)$ such that $W(t,x;z) = J^+(t,x;z)C(t;z)$. By the system (1.16), we calculate the time derivative of $C(t;z)$

$$\begin{aligned}
&\frac{d}{dt}C(t;z) = \frac{d}{dt}(J^+(t,x;z)^{-1}W(t,x;z)) \\
&= J^+(t,x;z)^{-1}\partial_t W(t,x;z) - J^+(t,x;z)^{-1}\partial_t J^+(t,x;z)J^+(t,x;z)^{-1}W(t,x;z) \\
&= J^+(t,x;z)^{-1}V(t,x;z)J^+(t,x;z)C(t;z) - J^+(t,x;z)^{-1}\partial_t J^+(t,x;z)C(t;z).
\end{aligned}$$

Since for all the time t,

$$J^+(t,x;z) \to \begin{pmatrix} e^{-izx} & 0 \\ 0 & e^{izx} \end{pmatrix}, \quad V(t,x;z) \to -2iz^2\sigma_3 \text{ as } x \to +\infty,$$

and $C(t;z)$ does not depend on x-variable, we deduce that

$$\frac{d}{dt}C(t;z) = -2iz^2\sigma_3 C(t;z),$$

and hence a particular solution can be $C(t;z) = e^{-2iz^2t\sigma_3}$ and $W^+(t,x;z) = J^+(t,x;z)e^{-2iz^2t\sigma_3}$ is the fundamental solution of the compatible systems (1.15)-(1.16).

B The NLS hierarchy

This section can be found in [Palais 1997 Bulletin AMS]. Recall the Lax equation (1.22) (for the defocusing case):

$$\psi_x = \begin{pmatrix} -iz & u \\ \bar{u} & iz \end{pmatrix} \psi := U\psi. \tag{B.28}$$

We can take the standard 2×2 Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\sigma_+ = \frac{\sigma_1 + i\sigma_2}{2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- = \frac{\sigma_1 - i\sigma_2}{2} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

to rewrite

$$U = U_0 + zU_1, \quad U_0 = u\sigma_+ + \bar{u}\sigma_-, \quad U_1 = \frac{1}{i}\sigma_3.$$

We search for equations

$$\partial_t \psi = V(z,y)\psi, \tag{B.29}$$

such that the compatibility condition between (B.28) and (B.29) is an equation for u. More precisely we will search for V such that

$$[\partial_t - V, \partial_x - U] = \begin{pmatrix} 0 & \partial_t u - f \\ \overline{\partial_t u - f} & 0 \end{pmatrix} = 0.$$

We make an Ansatz

$$V(z,u) = \sum_{j=0}^{k} z^{k-j} V_j(u),$$

and it reduces to search for V_j satisfying

$$V_0 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \quad -\partial_x V_j - [V_j, U_0] - [V_{j+1}, U_1] = 0, \quad 0 < j < k.$$

We write $V_j = \begin{pmatrix} -ir_j & p_j \\ \bar{p}_j & ir_j \end{pmatrix}$ and obtain the following recursive formulas:

$$p_{j+1} = \frac{i}{2}\partial_x p_j + ur_j, \quad r_j' = i(\bar{p}_j u - p_j \bar{u}),$$

together with the equation for u:

$$i\partial_t u = 2p_{k+1}.$$

This implies immediately

$$\begin{aligned}
&(p_0, r_0) = (0, 1), \quad (p_1, r_1) = (u, 0), \quad (p_2, r_2) = (\frac{i}{2}u', \frac{1}{2}|u|^2),\\
&(p_3, r_3) = \big(-\frac{1}{4}u'' + \frac{1}{2}u|u|^2, \frac{i}{4}(u'\bar{u} - u\bar{u}')\big),\\
&(p_4, r_4) = \big(-i\frac{1}{8}u''' + i\frac{3}{4}u'|u|^2, -\frac{1}{8}(|u|^2)'' + \frac{3}{8}|u'|^2 + \frac{3}{8}|u|^4\big), \cdots
\end{aligned}$$

and the corresponding equation (B.29) read as

$$\begin{aligned}
\partial_t \psi &= \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \psi,\\
\partial_t \psi &= \begin{pmatrix} -iz & u \\ \bar{u} & iz \end{pmatrix} \psi,\\
\partial_t \psi &= \begin{pmatrix} -iz^2 - \frac{i}{2}|u|^2 & zu + \frac{i}{2}u' \\ z\bar{u} - \frac{i}{2}\bar{u}' & iz^2 + \frac{i}{2}|u|^2 \end{pmatrix} \psi,\\
\partial_t \psi &= \begin{pmatrix} -iz^3 - \frac{iz}{2}|u|^2 + \frac{1}{4}(u'\bar{u} - u\bar{u}') & z^2u + \frac{iz}{2}u' - \frac{1}{4}u'' + \frac{1}{2}|u|^2u \\ z^2\bar{u} - \frac{iz}{2}\bar{u}' - \frac{1}{4}\bar{u}'' + \frac{1}{2}|u|^2\bar{u} & iz^3 + \frac{iz}{2}|u|^2 - \frac{1}{4}(u'\bar{u} - u\bar{u}') \end{pmatrix} \psi,\\
\partial_t \psi &= \begin{pmatrix} -iz^4 - \frac{iz^2}{2}|u|^2 + \frac{z}{4}(u'\bar{u} - u\bar{u}') + i\frac{1}{8}(|u|^2)'' - i\frac{3}{8}|u'|^2 - i\frac{3}{8}|u|^4 := A & \bar{B} \\ z^3\bar{u} - \frac{iz^2}{2}\bar{u}' - \frac{z}{4}\bar{u}'' + \frac{z}{2}|u|^2\bar{u} + i\frac{1}{8}u''' - i\frac{3}{4}u'|u|^2 := B & -A \end{pmatrix} \psi, \cdots
\end{aligned}$$

where in the above we assume $z \in \mathbb{R}$ when we take the complex conjugate of A, B. We see that (1.16) is the third equation in the above hierarchy and (NLS) reads as $i\partial_t u = 2p_3$ (up to factors).

It is true though not obvious that the compatibility conditions are Hamiltonian equations associated to the Hamiltonians

$$H_k = -\frac{1}{k+1}\int_{\mathbb{R}} tr\Big[V_{k+2}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}\Big]\,\mathrm{d}x = \frac{2}{k+1}\int_{\mathbb{R}} r_{k+2}\,\mathrm{d}x\,.$$

The Hamiltonian for the NLS flow $i\partial_t u = 2p_3$ hence reads as

$$H_2 = \frac{1}{4}\int_{\mathbb{R}} (|u'|^2 + |u|^4)\,\mathrm{d}x\,,$$

which is the conserved energy E defined in (1.10) (up to factors). The nonlinear Hamiltonian flows in the same hierarchy commute.

We obtain the equations for the focusing NLS hierarchy by replacing $\bar{u}$ by $-\bar{u}$ and the KdV hierarchy by replacing $\bar{u}$ by 1.

C Conserved energies

In this section we explain briefly the existence of a family of conserved energies for the one dimensional defocusing cubic nonlinear Schrödinger equation (NLS) (with $d = 1$, $p = 3$, $\kappa = 1$) established by [Koch-Tataru 2016] (see also Killip-Visan-Zhang). Recall that for the initial data $u_0 \in \mathcal{S}(\mathbb{R})$, there exists a unique solution $u \in C(\mathbb{R}; \mathcal{S}(\mathbb{R}))$ solving the 1-d cubic nonlinear Schrödinger equation (NLS) (see Remark 2.11), and the transmission coefficient $T = T(z)$ associated to the Lax equation (1.22) is invariant by the cubic Schrödinger flow (see (1.25)). We make use of this fact to establish the family of conserved energies $\mathcal{E}_s = \mathcal{E}_s(u)$ in terms of $T = T(z; u)$ which is equivalent to the Sobolev norm of the solution $\|u\|^2_{H^s}$, $\forall s > -\frac{1}{2}$.

Step 1 Expansions of the "transmission coefficient" $T^{-1}(z)$ and its logarithm $\ln T^{-1}(z)$ on the upper half plane

Recall the direct scattering transform introduced in Subsection 1.2. If $u(t,x) \in C(\mathbb{R}; \mathcal{S}(\mathbb{R}))$ is the solution of the one dimensional defocusing cubic nonlinear Schrödinger equation (NLS), $z = \xi \in \mathbb{R}$ (the continuous spectrum of the self-adjoint Lax operator), then we have the Jost solution $j^{-,1}(x; z)$ of the Lax equation (1.22)

$$\psi_x = \begin{pmatrix} -iz & u \\ \bar{u} & iz \end{pmatrix} \psi,$$

with the asymptotic behaviours at infinity:

$$j^{-,1}(x;z) = \begin{pmatrix} e^{-izx} \\ 0 \end{pmatrix} + o(1) \text{ as } x \to -\infty,$$
$$j^{-,1}(x;z) = T^{-1}(z) \begin{pmatrix} e^{-izx} \\ 0 \end{pmatrix} + R(t,z)T^{-1}(z) \begin{pmatrix} 0 \\ e^{izx} \end{pmatrix} + o(1) \text{ as } x \to +\infty.$$

Hence the "transmission coefficient" $T^{-1}(z)$ can be calculated as the first component of the renormalised Jost solution $l^{-,1}(x;z) = e^{izx} j^{-,1}(x;z)$ as

$x \to \infty$ (recalling the iterative formulation of $l^{-,1}$ in Subsection 1.2.1):

$$T^{-1}(z) = \lim_{x\to\infty}(e^{izx}j^{-,1}(x;z)) = 1 + \frown + \frown\frown + \frown\frown\frown + \frown\frown\frown\frown + \cdots := 1 + \sum_{j=1}^{\infty} T_{2j}(z), \tag{C.30}$$

where the graphic symbols $\frown, \frown\frown, \cdots$ denote the multilinear integrals as

$$\frown = T_2(z) = \int_{x<y} e^{2iz(y-x)}u(y)\bar{u}(x)\,\mathrm{d}x\;\mathrm{d}y\,,$$

$$\frown\frown = T_4(z) = \int_{x_1<y_1<x_2<y_2} \prod_{n=1}^{2} e^{2iz(y_n-x_n)}u(y_n)\bar{u}(x_n)\,\mathrm{d}x\;\mathrm{d}y\,, \quad \cdots$$

For any $z \in \overline{\mathcal{U}}, \mathcal{U} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ the upper half plane, we can define the Jost solution $j^{-,1}(x;z)$ and this provides a holomorphic extension of $T^{-1}(z)$ to the closed upper half plane $z \in \overline{\mathcal{U}}$, such that $|T| \leqslant 1$ on the real line and $T \to 1$ at infinity. By maximum principle, $|T^{-1}| \geqslant 1$ on $\overline{\mathcal{U}}$.

Since ln is analytic near 1 and expands formally as $\ln(1+a) = a - \frac{1}{2}a^2 + \frac{1}{3}a^3 + \cdots$, the formal series for T^{-1} yields a formal series for $\ln T^{-1}$:

$$\ln T^{-1} = \Big(T_2\Big) + \Big(T_4 - \frac{1}{2}(T_2)^2\Big) + \Big(T_6 - T_2T_4 + \frac{1}{3}(T_2)^2\Big) + \cdots := \sum_{j=1}^{\infty} \tilde{T}_{2j},$$

where each $\tilde{T}_{2j}(z)$ is homogeneous of degree $2j$ in terms of $\{u, \bar{u}\}$ and is a linear combination of multilinear integrals of the form

$$\int_{\Sigma_{2j}} \prod_{n=1}^{j} e^{2iz(y_n-x_n)}u(y_n)\bar{u}(x_n)\,\mathrm{d}x\;\mathrm{d}y\,,$$

with Σ_{2j} denoting an ordering obeying the constraint $\{x_n < y_n,\ \forall 1 \leqslant n \leqslant j\}$.

We can calculate for example that

$$\begin{aligned}(T_2)^2 &= \Big(\int_{x_1<y_1} e^{2iz(y_1-x_1)}u(y_1)\bar{u}(x_1)\,\mathrm{d}x_{\,1}\,\mathrm{d}y_{\,1}\Big)\Big(\int_{x_2<y_2} e^{2iz(y_2-x_2)}u(y_2)\bar{u}(x_2)\,\mathrm{d}x_{\,2}\,\mathrm{d}y_{\,2}\Big)\\ &= 4\int_{x_1<x_2<y_1<y_2} \prod_{n=1}^{2} e^{2iz(y_n-x_x)}u(y_n)\bar{u}(x_n)\,\mathrm{d}x\;\mathrm{d}y\\ &\quad + 2\int_{x_1<y_1<x_2<y_2} \prod_{n=1}^{2} e^{2iz(y_n-x_x)}u(y_n)\bar{u}(x_n)\,\mathrm{d}x\;\mathrm{d}y \;:= 4\overset{\frown}{\frown} + 2\frown\frown,\end{aligned}$$

which is homogeneous of degree 4 in $\{u, \bar{u}\}$, and the first terms in the expansion of $\ln T^{-1}$ read as

$$\ln T^{-1}(z) = \sum_{j=1}^{\infty} \tilde{T}_{2j}(z) = \frown + (-2\overset{\frown}{\frown}) + (12\overset{\frown}{\overset{\frown}{\frown}} + 4\overset{\frown}{\frown\frown}) + \cdots \tag{C.31}$$

Indeed, we can prove that $\tilde{T}_{2j}$ is a linear combination of *connected* symbols, that is, Σ_{2j} represents a complete ordering $\{x_1 < x_2 < \cdots < y_1 \,|\, x_n < y_n,\, 1 \leqslant n \leqslant j\}$ where the first and the last arcs are connected. To this end we introduce a Hopf algebra structure:

- We consider the set $H^{\sqcup\!\sqcup}$ of the words in the alphabet {◜, ◝} and introduce the shuffle product $\sqcup\!\sqcup$ between two words as the sum of all the words obtained by shuffling these two words such that $H^{\sqcup\!\sqcup}$ becomes a ring of formal power series with the words as unknowns, with the shuffle product defining the commutative, associative and distributive multiplication;

- We restrict ourselves in the subalgebra H consisting of nonintersecting words and the length $2j$ of the nonintersecting words in H introduces a grading on H such that words with length $2j$ have degree j and $H_m = H/I_m$ is a finite dimensional algebra where I_m is the ideal consisting of the elements of degree $> m$;

- We introduce a coproduct $\Delta : H \mapsto H \otimes H$ on H as the sum of all the splittings $\Delta a = \sum_{a_1 a_2 = a, a_1, a_2 \in H} a_1 \otimes a_2$, $\forall a \in H$, then the coproduct is coassociative which also satisfies the compatibility condition $\Delta(a \sqcup\!\sqcup b) = \Delta a \sqcup\!\sqcup \Delta b$ with the shuffle product of the tensor product defined as $(a \otimes b) \sqcup\!\sqcup (c \otimes d) = (a \sqcup\!\sqcup c) \otimes (b \sqcup\!\sqcup d)$;

- We call an element $g \in H$ group-like if $\Delta g = g \otimes g$, then the set of all group-like elements in H endowed with the shuffle product becomes a group G. On the other side, we call an element $p \in H$ primitive if $\Delta p = 1 \otimes p + p \otimes 1$, then the primitive element is formal linear combinations of connected symbols and all the primitive elements form a subspace $P \subset H$. The subgroup G and the subspace P are related by $G = \exp P$;

- In conclusion, the element T^{-1} (C.30) is a group-like element and hence its logarithm $\ln T^{-1}$ (C.31) is a primitive element such that its homogeneous parts $\tilde{T}_{2j}$ are linear combinations of connected symbols.

We are now interested in two issues:

- We would like to show the convergence of the expansions (C.30)-(C.31), and we are in particular interested in the decay rates of the terms $T_{2j}(z), \tilde{T}_{2j}(z)$ as $z \to i\infty$, when $u(x) \in H^s(\mathbb{R})$. This will be done in Step 3.

Roughly speaking, for $u \in \mathcal{S}(\mathbb{R})$, if $z = i\tau/2$, the bulk of the integrals in $T_{2j}(z)$, $\tilde{T}_{2j}(z)$ comes from the regions $\{y_n - x_n \lesssim \tau^{-1},\ 1 \leqslant n \leqslant j\}$, and in the case of $T_{2j}(z)$ the volume of the region is comparable to τ^{-j} while in the case of $\tilde{T}_{2j}(z)$ the volume of the region is comparable to $C(j)\tau^{-2j+1}$ which is much smaller than τ^{-j} if $j \geqslant 2$. Hence for $u(x) \in \mathcal{S}(\mathbb{R})$ we expect to have the expansion of $\ln T^{-1}$ in the powers of z^{-1} at infinity:

$$\ln T^{-1}(z) = i \sum_{j=1}^{\infty} \mathcal{E}_j (2z)^{-j}, \text{ as } |z| \to \infty,$$

where the (invariant) coefficients $\mathcal{E}_j$ are called energies and in particular

$$\begin{aligned}
\mathcal{E}_1 &= \int_{\mathbb{R}} |u|^2 \,\mathrm{d}x\,,\\
\mathcal{E}_2 &= -\mathrm{Im} \int_{\mathbb{R}} u'\bar{u} \,\mathrm{d}x\,,\\
\mathcal{E}_3 &= \int_{\mathbb{R}} (|u'|^2 + |u|^4) \,\mathrm{d}x\,,
\end{aligned}$$

are the (rescaled) mass, momentum and energy M, P, E defined in (1.8), (1.9), (1.10) respectively. We will indeed be interested in the case $u \in H^s$ and study the expansions (C.30), (C.31) by establishing the decay rates for $T_{2j}(z), \tilde{T}_{2j}(z)$ as $|z| \to \infty$ in terms of $\|u\|_{H^s}^2$.

- We would like to describe $\|u\|_{H^s}^2$ in terms of $T^{-1}(z)$, which is done in Step 2.

Step 2 Description of $\|u\|_{H^s}^2$ in terms of the quadratic term $T_2(z) = \tilde{T}_2(z) = \cap$

Let $z \in \mathcal{U}$ and let us calculate the quadratic term $\cap$ and relate it to the Fourier transform of u: $\hat{u}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi x} u(x)\,\mathrm{d}x$ as

$$\cap = \frac{1}{2\pi} \int_{\mathbb{R}^3} \int_{-\infty}^{y} e^{2iz(y-x)} e^{iy\eta} \hat{u}(\eta) e^{-ix\xi} \bar{\hat{u}}(\xi) \,\mathrm{d}x \,\mathrm{d}y \,d\xi d\eta,$$

where integration by parts in x implies

$$\cap = \frac{1}{2\pi} \int_{\mathbb{R}^3} \frac{-1}{2iz + i\xi} e^{iy\eta - ix\xi} \hat{u}(\eta) \bar{\hat{u}}(\xi) \,\mathrm{d}y \,d\xi d\eta = \int_{\mathbb{R}} \frac{i}{2z + \xi} \hat{u}(\xi) \bar{\hat{u}}(\xi) d\xi,$$

and hence

$$\frac{1}{\pi} \mathrm{Re} \cap (z/2) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{\mathrm{Im}\, z}{|z - \xi|^2} |\hat{u}(\xi)|^2 \,\mathrm{d}\xi \ := \frac{1}{\pi} \int_{\mathbb{R}} \frac{\mathrm{Im}\, z}{|z - \xi|^2} d\mu(\xi), \qquad \text{(C.32)}$$

which is the harmonic function on the upper half plane with the trace measure $d\mu = |\hat{u}(\xi)|^2\,\mathrm{d}\xi$. Hence by a change of integral contour we have the following description of the H^s, $-\frac{1}{2} < s < 0$-norm of u:

$$\begin{aligned}\|u\|_{H^s}^2 &= \int_{\mathbb{R}} (1+\xi^2)^s |\hat{u}(\xi)|^2\,\mathrm{d}\xi \\ &= -2\sin(\pi s)\int_1^\infty (\tau^2-1)^s \frac{1}{\pi}\mathrm{Re}\,\wedge(i\tau/2)d\tau.\end{aligned} \tag{C.33}$$

For general $u \in H^s$ with $N = [s] \geqslant 0$, we have the finite expansion for the above harmonic function evaluated at $z = i\tau$ with $\tau \to \infty$:

$$\frac{1}{\pi}\mathrm{Re}\,\wedge(i\tau/2) = \sum_{l=0}^{N}(-1)^l H_{2l}\tau^{-2l-1} + (-1)^{N+1}H_{>2N}(\tau)\tau^{-2N-1},$$

where

$$H_{2l} = \frac{1}{\pi}\int_{\mathbb{R}} \xi^{2l} d\mu(\xi), \quad H_{>2N}(\tau) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{\xi^{2N+2}}{\tau^2+\xi^2} d\mu(\xi).$$

Then the description of the Sobolev norm $\|u\|_{H^s}^2$ in terms of $\wedge$ reads as

$$\begin{aligned}\|u\|_{H^s}^2 = -2\sin(\pi s)\int_1^\infty (\tau^2-1)^s\Big(\frac{1}{\pi}\mathrm{Re}\,\wedge(i\tau/2) - \sum_{l=0}^{N}(-1)^l H_{2l}\tau^{-2l-1}\Big)d\tau \\ + \pi\sum_{l=0}^{N}\binom{s}{l}H_{2l}.\end{aligned} \tag{C.34}$$

Step 3 Well-definedness of the "transmission coefficient" $T^{-1}(z)$ if $u \in H^s$ and the estimates for higher order terms $T_{2j}(z)$, $\tilde{T}_{2j}(z)$

We introduce the functional spaces U^2, V^2 and DU^2, as substitutes for $\dot{H}^{\frac{1}{2}}$ and $\dot{H}^{-\frac{1}{2}}$, which are more suitable for the estimates for the integrals in $T_{2j}(z), \tilde{T}_{2j}(z)$. We define V^2 as the space of the functions v on $\mathbb{R}$ such that the following norm is finite

$$\|v\|_{V^2} = \sup_{-\infty<t_1<\cdots<t_L=+\infty}\Big(\sum_{j=1}^{L-1}|v(t_{j+1})-v(t_j)|^2\Big)^{\frac{1}{2}}, \text{ where we set } v(\infty)=0.$$

We call the step function of form $\sum_{j=1}^{L-1}\phi_j \mathbf{1}_{[t_j,t_{j+1})}$ with $\sum_j |\phi_j|^2 = 1$ a U^2 atom and a U^2 function u is an ℓ^1-sum of U^2 atoms such that $\|u\|_{U^2} = \inf_{(\lambda_k)}\{\sum_k |\lambda_k| \mid u = \sum_k \lambda_k u_k,\ (\lambda_k) \in \ell^1(\mathbb{N}) \text{ and } u_k \text{ are } U^2\text{-atoms}\} < \infty$. We have the following pleasant properties concerning V^2, U^2-norms:

- The V^2-functions v have left and right limits everywhere and $v(\infty) = 0$ such that $\|v\|_{L^\infty} \leqslant \|v\|_{V^2}$ and obviously $\|uv\|_{V^2} \leqslant \|u\|_{V^2}\|v\|_{L^\infty} + \|u\|_{L^\infty}\|v\|_{V^2}$;
- The U^2-functions u are right continuous and $u(-\infty) = 0$ such that $\|u\|_{V^2} \leqslant \sqrt{2}\|u\|_{U^2}$ and we have the embedding estimates $\|u\|_{B^{\frac{1}{2}}_{2,\infty}} \lesssim \|u\|_{V^2} \lesssim \|u\|_{U^2} \lesssim \|u\|_{B^{\frac{1}{2}}_{2,1}}$;
- There exists a bilinear form $B(v,u)$ which induces an isometric isomorphism from $V^2 \mapsto (U^2)^*$ and from $U^2 \mapsto (V^2_C)^*$, $V^2_C = \{v \in V^2 \,|\, v \text{ is continuous }, v(\pm\infty) = 0\}$, such that [5]

$$\begin{aligned}\|v\|_{V^2} &= \sup{}_{u\in U^2}\{B(v,u) \,|\, \|u\|_{U^2} = 1\},\\ \|u\|_{U^2} &= \sup{}_{v\in V^2}\{B(v,u) \,|\, \|v\|_{V^2} = 1\},\end{aligned}$$

  where by an abuse of notation $B(v,u) = \int_{\mathbb{R}} v du$.

We introduce DU^2 functions as the distributional derivatives of U^2 functions: $DU^2 = \{u' \,|\, u \in U^2\}$ and hence DU^2 function is distribution function with the finite norm $\|f\|_{DU^2} = \sup\{\int_{\mathbb{R}} fv\,\mathrm{d}x \mid \|v\|_{V^2} \leqslant 1\}$. Thus we have

$$\|uv\|_{DU^2} \leqslant 2\|u\|_{DU^2}\|v\|_{V^2} \leqslant 4\|u\|_{DU^2}\|v\|_{U^2}.$$

For any $z \in \mathcal{U}$, we define the one step operator A as

$$A(\phi)(t) = \int_{x<y<t} e^{2iz(y-x)}u(y)\bar{u}(x)\phi(x)\,\mathrm{d}x\,\mathrm{d}y\,,$$

then the terms in the expansion (C.30): $T^{-1} = 1 + \sum_{j=1}^\infty T_{2j}$ read as

$$T_2 = \lim_{t\to\infty} A(1)(t), \quad T_4 = \lim_{t\to\infty} A^2(1)(t), \quad T_{2j} = \lim_{t\to\infty} A^j(1)(t), \cdots$$

and T_{2j} can be bounded by the operator norm $\|A\|_{V^2\mapsto U^2}$ as follows:

$$|T_{2j}| \leqslant 2\|A^j(1)\|_{U^2} \leqslant 2\|A\|_{V^2\mapsto U^2}\|A^{j-1}(1)\|_{V^2} \leqslant \cdots \leqslant 2^j\|A\|^j_{V^2\mapsto U^2}. \quad \text{(C.35)}$$

We hence estimate the operator norm $\|A\|_{V^2\mapsto U^2}$ as follows:

$$\begin{aligned}&\|A(\phi)\|_{U^2} = \Big\|\int_{x<y} e^{2iz(y-x)}u(y)\bar{u}(x)\phi(x)\,\mathrm{d}x\Big\|_{DU^2_y}\\ &\leqslant 4\|e^{2i\mathrm{Re}\,zy}u\|_{DU^2_y}\,\|\varphi * (e^{-2i\mathrm{Re}\,z\cdot}\bar{u}\phi)\|_{U^2_y}, \quad \varphi = e^{-2\mathrm{Im}\,zt}\mathbf{1}_{t>0}\\ &\leqslant 4\|e^{2i\mathrm{Re}\,zy}u\|_{DU^2}\,\|e^{-2i\mathrm{Re}\,z\cdot}\bar{u}\phi\|_{DU^2} \leqslant 8\|e^{2i\mathrm{Re}\,zy}u\|_{DU^2}\,\|e^{-2i\mathrm{Re}\,z\cdot}\bar{u}\|_{DU^2}\|\phi\|_{V^2},\end{aligned}$$

[5] We can restrict the supermum to step functions. If $\sup\{B(v,u) \,|\, \|u\|_{U^2} = 1\} < \infty$, then $v \in V^2$, and if $\sup\{B(v,u) \,|\, \|v\|_{V^2} = 1\} < \infty$ and u is right continuous ruled function with $\lim_{t\to-\infty} u(t) = 0$, then $u \in U^2$.

where we used $\|\varphi * f\|_{U^2} = \|\varphi' * f\|_{DU^2} \leqslant \|f\|_{DU^2}$, and hence

$$\|A\|_{V^2 \mapsto U^2} \leqslant 8\|e^{2i\mathrm{Re}\, z \cdot} u\|_{DU^2}^2.$$

We have indeed better estimates for $\mathrm{Im}\, z = \tau/2 > 0$: We can introduce the localised version of DU^2-norms:

$$\|u\|_{l^q_\tau DU^2} = \big\| \|\chi_{[\frac{k}{\tau}, \frac{k+1}{\tau}]} u\|_{DU^2} \big\|_{\ell^q_k},$$

where $(\chi_{[\frac{k}{\tau}, \frac{k+1}{\tau}]})_k$ form a partition of unity, such that

$$\|A\|_{V^2 \mapsto U^2} \leqslant C\|e^{2i\mathrm{Re}\, z \cdot} u\|_{l^2_\tau DU^2}^2 \leqslant C\frac{1 + \mathrm{Re}\, z + \tau}{\tau} \|u\|_{l^2_1 DU^2}^2, \tag{C.36}$$

where $l^2_1 DU^2$ can be seen as a substitute of $H^{-\frac{1}{2}}$ such that $H^{s_0} \hookrightarrow l^2_1 DU^2$ whenever $s_0 > -\frac{1}{2}$.

Therefore if $\|u\|_{l^2_1 DU^2} \ll 1$, then by virtue of (C.35)-(C.36) the formal series of $T^{-1} = 1 + \sum_{j \geqslant 1} T_{2j}$ converges in the region $\{z \in \mathcal{U} \mid \mathrm{Im}\, z \geqslant 1 + |\mathrm{Re}\, z|\}$ and in particular on the half-line $i[1, \infty)$. And generally, if $u \in l^2_1 DU^2$, then for any $z \in \mathcal{U}$, there exist $x_0, x_1 \in \mathbb{R}$ such that

$$\sqrt{(1 + \mathrm{Re}\, z + \mathrm{Im}\, z)/\mathrm{Im}\, z} \big(\|u|_{(-\infty, x_0]}\|_{l^2_1 DU^2} + \|u|_{[x_1, \infty)}\|_{l^2_1 DU^2}\big) \leqslant 1/2C \ll 1,$$

and hence we solve the Lax equation (1.22) first until x_0 (the solvability is ensured by the above argument for the small norm case), and then solve the Lax equation from x_0 to x_1 (the solvability on this finite interval $[x_0, x_1]$ is ensured by $u|_{[x_0, x_1]} \in DU^2$ and hence the Lipschitz property of the linear flow), and finally solve the Lax equation from x_1 to ∞ (the solvability is also ensured by the smallness), such that we can still define the holomorphic function $T^{-1}(z)$ as the first component of the solution at infinity.

In particular if $z = i\tau/2$, $\tau \geqslant 1$, then we have the following estimates for $T_{2j}, \tilde{T}_{2j}$, $j \geqslant 2$ similar as in (C.35)-(C.36):

$$|T_{2j}| \leqslant (C\|u\|_{l^2_\tau DU^2})^{2j}, \quad |\tilde{T}_{2j}| \leqslant C(j)\|u\|_{l^{2j}_\tau DU^2}^{2j}. \tag{C.37}$$

Step 4 Conserved energies

Motivated by the formulations (C.33)-(C.34) of the H^s-norm, we define the conserved energy via

$$\mathcal{E}_s(u) = -\frac{2\sin(\pi s)}{\pi} \int_1^\infty (\tau^2 - 1)^s G(i\tau/2)\, \mathrm{d}\tau\,, \ -\frac{1}{2} < s < 0, \tag{C.38}$$

and for general $N = [s] \geqslant 0$,

$$\begin{aligned}\mathcal{E}_s(u) = &-\frac{2\sin(\pi s)}{\pi}\int_1^\infty (\tau^2-1)^s\Big(G(i\tau/2) - \sum_{l=0}^N (-1)^l G_{2l}\tau^{-2l-1}\Big)\,\mathrm{d}\tau \\ &+ \pi\sum_{l=0}^N \binom{s}{l} G_{2l},\end{aligned} \tag{C.39}$$

where $G(z) = \mathrm{Re}\,\ln T^{-1}(z)$ is a nonnegative harmonic function (as the real part of the logarithm of the holomorphic function T^{-1} with $|T^{-1}| \geqslant 1$) on the upper half plane $\mathcal{U}$ and G_{2l} are the real coefficients of the expansion of $G(z)$ at infinity.

For $-\frac{1}{2} < s < 0$, we want to bound the following difference

$$|\mathcal{E}_s(u) - \|u\|_{H^s}^2| = \Big|\frac{2\sin(\pi s)}{\pi}\int_1^\infty (\tau^2-1)^s \mathrm{Re}\sum_{j=2}^\infty \tilde{T}_{2j}(i\tau/2)\,\mathrm{d}\tau\Big|.$$

Recall the estimate (C.37) and we now want to bound the above difference in terms of $\|u\|_{H^s}$: Let $z = i\tau/2$, then by use of the Littlewood-Paley decomposition $u = \sum_\mu u_\mu$, $\mu = 2^j$, $j = 0, 1, \cdots$ such that $\|u_\mu\|_{H^s} \sim d_\mu\|u\|_{H^s}$, $\|d_\mu\|_{\ell^2} = 1$,

$$\begin{aligned}\int_1^\infty (\tau^2-1)^s|T_2(i\tau/2)|\,\mathrm{d}\tau &\leqslant \int_1^\infty (\tau^2-1)^s\sum_{\mu,\nu}|A(1;\bar{u}_\mu,u_\nu)(i\tau/2)|\,\mathrm{d}\tau \\ \lesssim \int_1^\infty (\tau^2-1)^s &\sum_{\mu,\nu}\|u_\mu\|_{l^2_\tau DU^2}\|u_\nu\|_{l^2_\tau DU^2}\,\mathrm{d}\tau \\ \lesssim \int_1^\infty (\tau^2-1)^s &\sum_{\mu\geqslant\nu\geqslant\tau} d_\mu\mu^{-\frac{1}{2}-s}d_\nu\nu^{-\frac{1}{2}-s}\|u\|_{H^s}^2\,\mathrm{d}\tau + \sum_{\mu\geqslant\tau\geqslant\nu}\cdots + \sum_{\tau\geqslant\mu\geqslant\nu}\cdots \\ \lesssim \sum_{\mu\geqslant\nu\geqslant\tau} d_\mu(\frac{\tau}{\mu})^{\frac{1}{2}+s} &d_\nu(\frac{\tau}{\nu})^{\frac{1}{2}+s}\|u\|_{H^s}^2 + \sum_{\mu\geqslant\tau\geqslant\nu}\cdots + \sum_{\tau\geqslant\mu\geqslant\nu}\cdots \lesssim \|u\|_{H^s}^2,\end{aligned}$$

and similarly (but more complicatedly) we can bound

$$\int_1^\infty (\tau^2-1)^s(|T_{2j}(i\tau/2)| + |\tilde{T}_{2j}(i\tau/2)|)\,\mathrm{d}\tau \leqslant \|u\|_{H^s}^2(C\|u\|_{l_1^2DU^2})^{2j-2}.$$

Hence if $\|u\|_{l_1^2DU^2} \ll 1$, then

$$|\mathcal{E}_s(u) - \|u\|_{H^s}^2| \leqslant C\|u\|_{l_1^2DU^2}^2\|u\|_{H^s}^2, \tag{C.40}$$

such that the conserved energy $\mathcal{E}_s(u)$ is equivalent to $\|u\|_{H^s}^2$. Therefore the Sobolev norm $\|u\|_{H^s}$ is "conserved" by the cubic nonlinear Schrödinger flow (if the solution exists). Indeed, if initially $\|u_0\|^2_{l^2_1 DU^2} \lesssim \|u_0\|^2_{H^s} \leqslant c_0 \ll 1$ such that $|\mathcal{E}_s(u_0) - \|u_0\|^2_{H^s}| \leqslant \frac{1}{2}\|u_0\|^2_{H^s}$ and $\frac{1}{2}\|u_0\|^2_{H^s} \leqslant \mathcal{E}_s(u_0) \leqslant 2\|u_0\|^2_{H^s} \leqslant 2c_0$, then by the conservation law $\mathcal{E}_s(u) = \mathcal{E}_s(u_0)$ (if the solution u exists globally in time) and a bootstrap argument, we have (C.40) and the smallness condition $\|u\|^2_{l^2_1 DU^2} \lesssim \|u\|^2_{H^s} \leqslant 2\mathcal{E}_s(u) = 2\mathcal{E}_s(u_0) \leqslant 4c_0 \ll 1$ for all the times. For general initial data with $\|u_0\|_{H^s} = R < \infty$, we can do scaling $u_{0,\lambda}(x) = \frac{1}{\lambda}u_0(\frac{x}{\lambda})$ as in Subsection 1.1.3, such that for λ large enough

$$\|u_{0,\lambda}\|_{H^s} \lesssim \|u_{0,\lambda}\|_{\dot{H}^s} = \lambda^{-(s+\frac{1}{2})}R \leqslant c_0 \ll 1, \text{ if } s \in (-\frac{1}{2}, 0),$$

and for $s \geqslant 0$, we take λ such that $\lambda^{-2}R \leqslant c_0$. Finally we get the bound for $\|u\|_{H^s}$ from the control for $\|u_\lambda\|_{H^s}$.

For general $s \geqslant 0$, we can also derive the difference estimate $|\mathcal{E}_s(u) - \|u\|^2_{H^s}| \leqslant C\|u\|^2_{l^2_1 DU^2}\|u\|^2_{H^s}$, nevertheless we have to consider the finite expansions of the terms $\tilde{T}_{2j}$, $j < 2s + 1$ (similar as the finite expansion in the integrand in (C.34) for $T_2 = \tilde{T}_2 = \cap$) and we do not go further here.

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