Exercise 1

Suppose that $u(t,x) \in \mathcal{S}(\mathbb{R} \times \mathbb{R})$ satisfies the cubic NonLinear Schrödinger equation:

$$i\partial_t u + \Delta u = 2|u|^2 u. \tag{NLS}$$

Show that the matrices

$$W(t,x;z) = J(t,x;z)\, e^{-2iz^2 t\sigma_3}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are simultaneously fundamental solutions of the compatible linear systems

$$\partial_x \psi = \begin{pmatrix} -iz & u \\ \overline{u} & iz \end{pmatrix} \psi = U\psi, \tag{1.1}$$

$$\partial_t \psi = \begin{pmatrix} -2z^2 - |u|^2 & -2izu + u_x \\ -2iz\overline{u} + \overline{u}_x & 2z^2 + |u|^2 \end{pmatrix} \psi = V\psi. \tag{1.2}$$

Here $J(t,x;z)$ is the fundamental solution of (1.1) with the boundary condition

$$\lim_{x \to -\infty} J\, e^{izx\sigma_3} = \mathbb{I}.$$

Exercise 2

Let $u(t,x) \in \mathcal{S}(\mathbb{R} \times \mathbb{R})$ and $T_2(z) = \int_{x_1 < x_2} e^{2iz(x_2 - x_1)} u(x_2)\overline{u(x_1)}\, dx_1\, dx_2$. Use the inverse Fourier transform $u(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\xi} \widehat{u}(\xi)\, d\xi$ to show

$$T_2(z) = \int_{\mathbb{R}} \frac{i}{2z + \xi}\, \widehat{u}(\xi)\, \overline{\widehat{u}(\xi)}\, d\xi.$$

Exercise 3

Let $u(t, \sqrt{2}x) = \sqrt{\rho(t,x)} e^{i\phi(t,x)}$ be the solution of the NonLinear Schrödinger equation:

$$i\partial_t u + \Delta u = \kappa |u|^{p-1} u, \quad \kappa = \pm 1. \tag{NLS}$$

Show that (ρ, v) with $v = \nabla\phi$ satisfy the following system

$$\begin{cases} \partial_t \rho + \mathrm{div}_x(\rho v) = 0, \\ \partial_t v + v \cdot \nabla_x v + \nabla_x(\kappa \rho^{\frac{p-1}{2}}) = \nabla_x(\dfrac{\Delta\sqrt{\rho}}{2\sqrt{\rho}}). \end{cases}$$

Hint: Compare the real and imaginary part of (NLS).

Exercise 4

Consider the Maxwell's equations:

$$\operatorname{curl}\mathcal{E} = -\partial_t\mathcal{B}, \quad \operatorname{curl}\mathcal{H} = \partial_t\mathcal{D}, \quad \operatorname{div}\mathcal{D} = 0, \quad \operatorname{div}\mathcal{B} = 0,$$

where $t \in \mathbb{R}$, $x \in \mathbb{R}^3$ are the time and space variables, $\mathcal{E}, \mathcal{H} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$, and $\mathcal{D}, \mathcal{B} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$. In vaccum, $\mathcal{D}$, $\mathcal{B}$ are related to $\mathcal{E}$, $\mathcal{H}$ by the constitutive relations

$$\mathcal{D} = \epsilon_0\mathcal{E}, \quad \mathcal{B} = \mu_0\mathcal{H}.$$

Show that the electric field satisfies the wave equation

$$\partial_t^2\mathcal{E}_j - c^2\Delta\mathcal{E}_j = 0, = 1, 2, 3,$$

where $c = (\epsilon_0\mu_0)^{-\frac{1}{2}}$.
Hint: Check that $\operatorname{curl}\operatorname{curl} = -\Delta + \nabla\operatorname{div}$ in $\mathbb{R}^3$.