Exercise 1

Recall the NonLinear Schrödinger equation

$$\begin{cases} i\partial_t u + \Delta u = \kappa |u|^{p-1} u, \quad \kappa = \pm 1 \\ u|_{t=0} = u_0(x). \end{cases} \tag{NLS}$$

Let $p = 1 + \frac{4}{d}$ be the $L^2(\mathbb{R}^d)$ critical exponent. Show that

(1) (NLS) is locally well-posed in $L^2(\mathbb{R}^d)$ such that for any $u_0 \in L^2(\mathbb{R}^d)$, there exists a unique solution

$$u \in X_T = \{u \in C([-T, T]; L^2(\mathbb{R}^d)) \,|\, u \in L^{p+1}_{t,x}([-T, T] \times \mathbb{R}^d)\},$$

where $T > 0$ depending on u_0, d and there exists a neighborhood U of u_0 such that the flow map $\Phi : L^2 \to X_T$ via $\Phi : u_0 \mapsto u$ is Lipschitz continuous;

(2) There exists a sufficiently small constant $\varepsilon_0 > 0$ depending on d such that if $\|u_0\|_{L^2_x} < \varepsilon_0$ then (NLS) is globally well-posed in $L^2(\mathbb{R}^d)$ and the unique solution belongs to

$$C(\mathbb{R}; L^2_x(\mathbb{R}^d)) \cap L^{p+1}_{t,x}(\mathbb{R} \times \mathbb{R}^d).$$

Hint: Step 1: Show that for any $\varepsilon > 0$, for any $u_0 \in L^2$, there exists a neighborhood U of u_0 and $T_0 > 0$ such that

$$\|S(t)v_0\|_{L^{p+1}_{t,x}([-T_0,T_0]\times\mathbb{R}^d)} \le \varepsilon, \quad \forall v_0 \in U.$$

Step 2: Show that the map Ψ

$$\Psi : u \mapsto \Psi(u) = S(t)u_0 - i\kappa \int_0^t S(t-t')\big(|u(t')|^{p-1}u(t')\big)dt' \tag{1}$$

has the fixed point in the space

$$\{u \in X_{T_0} \,|\, \|u\|_{L^\infty_t([-T_0,T_0];L^2_x(\mathbb{R}^d))} \le 2C\|u_0\|_{L^2_x},\ \|u\|_{L^{p+1}_{t,x}([-T_0,T_0]\times\mathbb{R}^d)} \le 2\varepsilon\},$$

for some sufficiently small ε depending on $\|u_0\|_{L^2_x}, d$ and T_0 depending on ε, u_0.

Step 3: Show that the flow map $\Phi : U \mapsto X_{T_0}$ is Lipschitz continuous.

Step 4: To Show the global well-posedness result in L^2 for small initial data $\|u_0\|_{L^2_x} \le \varepsilon_0$, we use the similarly estimates as in Step 2.

Exercise 2

Let $s > 0$ and

$$p_c = \begin{cases} \dfrac{2d}{d-2s}, & \text{if} \quad s < \dfrac{d}{2}, \\ \infty, & \text{if} \quad s \ge \dfrac{d}{2}. \end{cases}$$

(1) Take the smooth mollifier function: $\varphi \in C_0^\infty(\mathbb{R}^d)$, $\varphi \geq 0$, $\int_{\mathbb{R}^d} \varphi = 1$ and its rescaled functions $\varphi_\varepsilon(x) = \varepsilon^{-d}\varphi(\varepsilon^{-1}x)$. Show that

$$\sup_{\|g\|_{H^s}\leq 1} \|\varphi_\varepsilon * g - g\|_{L^2(\mathbb{R}^d)} \to 0 \text{ as } \varepsilon \to 0, \text{ if } s > 0,$$

$$\sup_{\|g\|_{H^s}\leq 1} \|\varphi_\varepsilon * g - g\|_{L^\infty(\mathbb{R}^d)} \to 0 \text{ as } \varepsilon \to 0, \text{ if } s > \frac{d}{2},$$

while for any $\varepsilon > 0$, $\sup_{\|g\|_{L^2}=1} \|\varphi_\varepsilon * g - g\|_{L^2(\mathbb{R}^d)} \geq 1$.

(2) For any fixed $\varepsilon > 0$, for any fixed $R > 0$, show that the map

$$\varphi_\varepsilon * : L^2(\mathbb{R}^d) \mapsto L^\infty(\bar{B}_R), \quad \bar{B}_R = \{x \in \mathbb{R}^d \| x| \leq R\}$$

is compact.
Hint: Use the Young's inequality and Arzela-Ascoli's theorem.

(3) Show that the identity map

$$\mathrm{Id} : H^s(\mathbb{R}^d)(\subset L^2(\mathbb{R}^d)) \mapsto L^2(\bar{B}_R)(\subset L^\infty(\bar{B}_R)), \quad s > 0$$

is compact.
Hint: Use that the identity map is the uniform limit of $\varphi_\varepsilon *$.

(4) Show that the embedding $H^s(\mathbb{R}^d) \hookrightarrow L^p_{loc}(\mathbb{R}^d)$, $1 \leq p < p_c$ if $s \leq \frac{d}{2}$ and $1 \leq p \leq p_c$ if $s > \frac{d}{2}$ are compact in the following sense: For any bounded sequence $(f_n)_n$ in $H^s(\mathbb{R}^d)$, there exists a subsequence $(f_{\psi(n)})_n$ and $f \in H^s(\mathbb{R}^d)$ such that for any compact set $K \subset\subset \mathbb{R}^d$

$$f_{\psi(n)} \to f \text{ in } L^p(K).$$

Hint: Use the Sobolev emberdding $H^s(\mathbb{R}^d) \hookrightarrow L^{p_c}(\mathbb{R}^d)$ if $s < \frac{d}{2}$. Then by Hölder's inequality to show $H^s(\mathbb{R}^d) \hookrightarrow\hookrightarrow L^p(\bar{B}_R)$ compactly for all $p \in [1, p_c)$ and Cantor's diagonal argument ensures the compact embedding $H^s(\mathbb{R}^d) \hookrightarrow\hookrightarrow L^p_{loc}(\mathbb{R}^d)$. Similar result holds for $s \geq \frac{d}{2}$.

Exercise 3

Let $s > 0$ and

$$p_c = \begin{cases} \dfrac{2d}{d-2s}, & \text{if } \quad s < \dfrac{d}{2}, \\ \infty, \quad \text{if} & s \geq \dfrac{d}{2}. \end{cases}$$

Show that the embeddings $H^s(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d)$, $2 \leq p < p_c$ and $H^s(\mathbb{R}^d) \hookrightarrow L^{p_c}_{loc}(\mathbb{R}^d)$ if $s < \frac{d}{2}$ are both not compact.

Exercise 4

Find a function in $H^1(\mathbb{R}^2)$ but not in $L^\infty(\mathbb{R}^2)$.