Exercise 1

Recall the NonLinear Schrödinger equation

$$\begin{cases} i\partial_t u + \Delta u = \kappa |u|^{p-1} u, \quad \kappa = \pm 1, \\ u|_{t=0} = u_0(x). \end{cases} \tag{NLS}$$

Let $1 < p < \infty$ if $d = 1, 2$ and $1 < p < 1 + \frac{4}{d-2}$ if $d \geq 3$ be a H^1 subcritical exponent. Let $u_0 \in H^1(\mathbb{R}^d)$.

(1) Show that (NLS) is locally well-posed in $H^1(\mathbb{R}^d)$: There exists a positive time $T > 0$ depending on $\|u_0\|_{H^1}, p, d$, a unique solution

$$u \in Y_T = \{u \in C([-T,T]; H^1(\mathbb{R}^d)) \,|\, u \in L^q([-T,T]; W^{1,\rho}(\mathbb{R}^d))\}$$

with admissible exponent pair (q, ρ), and there exists a neighborhood V of u_0 in H^1 such that the flow map

$$\Phi : V \to Y_T \text{ via } u_0 \mapsto u$$

is Lipschitz continuous.

(2) Let $\kappa = -1$ and $1 + \frac{4}{d} < p$. Show that there exists some sufficiently small constant ϵ_0, such that when $\|u_0\|_{H^1} \leq \varepsilon_0$ we have

$$u \in C(\mathbb{R}; H^1_x) \cap L^q(\mathbb{R}; W^{1,\rho}(\mathbb{R}^d)) \text{ with admissible exponent pair } (q, \rho). \tag{1}$$

Hint: Step 1: Show that the nonlinear map

$$\Psi : u \mapsto \Psi(u) = S(t)u_0 - i\kappa \int_0^t S(t-t')\big(|u(t')|^{p-1}u(t')\big)dt' \tag{2}$$

is well-defined and contractive in

$$Y_T(R) = \{u \in C([-T,T]; H^1) \cap L^q([-T,T]; W^{1,\rho}) | \|u\|_T = \|u\|_{L^\infty_T L^2} + \|u\|_{L^q_T W^{1,\rho}} \leq R\}$$

for appropriately chosen admissible exponent pair (q, ρ) with $q > p \geq \rho/2 > 1$ if $d = 1, 2$, $(q, \rho) = \big(\frac{4(p+1)}{(d-2)(p-1)}, \frac{d(p+1)}{d+p-1}\big)$ if $d \geq 3$ and R, T.

Step 2: Show that the flow map $\Phi : V \to Y_T$ is Lipschitz continuous.

Step 3: Show the global well-posedness result in (2).

Exercise 2

Show the Pohozaev's Identity

$$\begin{aligned} &\mathrm{Re} \int_{\mathbb{R}^d} \Delta \bar{u}\big(x \cdot \nabla u\big)\, dx = (\frac{d}{2} - 1) \int_{\mathbb{R}^d} |\nabla u|^2\, dx, \quad \forall u \in \mathcal{S}(\mathbb{R}^d), \\ &\text{or equivalently, } \mathrm{Re} \int_{\mathbb{R}^d} \Delta \bar{u}\big(\frac{d}{2}u + x \cdot \nabla u\big)\, dx = -\int_{\mathbb{R}^d} |\nabla u|^2\, dx, \end{aligned} \tag{3}$$

where $|\nabla u|^2 = \sum_{j=1}^d ((\partial_{x_j} \mathrm{Re}\, u)^2 + (\partial_{x_j} \mathrm{Im}\, u)^2)$.

Exercise 3

Let

$$u \in C([-T,T];H^1(\mathbb{R}^d)) \cap L^q([-T,T];W^{1,\rho}(\mathbb{R}^d)), \quad |u|^{p-1}u \in L^{q'}([-T,T];W^{1,\rho'}(\mathbb{R}^d)),$$

where (q,ρ) is an admissible pair.

Take $\varphi \in C_0^\infty(\mathbb{R}^d)$ with $\varphi \geq 0$, $\int_{\mathbb{R}^d} \varphi(x) = 1$. Denote $\varphi_n(x) = n^d \varphi(nx)$. Similarly we take $\psi \in C_0^\infty([-T,T])$, $\psi \geq 0$, $\int \psi(t)dt = 1$ and denote $\psi_m(t) = m\psi(mt)$. We denote $u_{m,n} = u * \varphi_n * \psi_m$.

Show that

$$u_{m,n} \to u \text{ in } L_T^q W^{1,\rho},$$
$$|u_{m,n}|^{p-1}u_{m,n},\ (|u|^{p-1}u)_{m,n} \to |u|^{p-1}u \text{ in } L_T^{q'}W^{1,\rho'} \cap L_T^{q'}L^{(\frac{1}{\rho'}-\frac{1}{d})^{-1}} \cap L_T^q L^{(\frac{1}{\rho}+\frac{1}{d})^{-1}}.$$

Hint: By the Sobolev embedding and the log-convexity of L^p norms ($\|f\|_{L^{p_\theta}} \leq \|f\|_{L^{p_0}}^{1-\theta}\|f\|_{L^{p_1}}^{\theta}$ if $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$) show that

$$u \in \left(L_T^\infty H^1 \cap L_T^q W^{1,\rho}\right) \subset \left(L_T^\infty L^{(\frac{1}{2}-\frac{1}{d})^{-1}} \cap L_T^q L^{(\frac{1}{\rho}-\frac{1}{d})^{-1}}\right) \subset L_T^{p\alpha} L^{p(\frac{1}{\rho}+\frac{1}{d})^{-1}},$$
$$|u|^{p-1}u \in L_T^{q'}W^{1,\rho'} \subset L_T^{q'}L^{(\frac{1}{\rho'}-\frac{1}{d})^{-1}},$$

for some $\alpha > q$.

Exercise 4

Recall the virial potential

$$V(u) = \int_{\mathbb{R}^d} |x|^2|u|^2\,dx, \tag{4}$$

and the associated Morawetz action

$$W(u) = \operatorname{Im}\sum_{j=1}^d \int_{\mathbb{R}^d} x_j(\bar{u}\partial_{x_j}u)\,dx \equiv \operatorname{Im}\int_{\mathbb{R}^d} r(\bar{u}\partial_r u)\,dx, \quad r = |x|. \tag{5}$$

Recall the Cauchy problem (NLS)

$$\begin{cases} i\partial_t u + \Delta u = \kappa|u|^{p-1}u, \quad \kappa = \pm 1, \\ u|_{t=0} = u_0(x). \end{cases} \tag{NLS}$$

(1) Let $u(t,x) \in \mathcal{S}(\mathbb{R}\times\mathbb{R}^d)$ be a solution of (NLS). Show that

$$\frac{1}{4}\frac{d}{dt}V(u(t)) = W(u(t)), \tag{6}$$

and

$$\frac{1}{2}\frac{d}{dt}W(u(t)) = \int_{\mathbb{R}^d} |\nabla u|^2 + \kappa\left(\frac{d}{2} - \frac{d}{p+1}\right)\int_{\mathbb{R}^d} |u|^{p+1}\,dx. \tag{7}$$

(2) Let $p \in (1, 2^*-1)$ be energy subcritical exponent. Let $u_0 \in \Sigma$ and let $u \in C([-T,T];H^1)$, $T < \infty$ be the solution of (NLS). Recall the operator $P = x + 2it\nabla$. Then $u \in C([-T,T];\Sigma) \cap L^q([-T,T];W^{1,\rho})$, $Pu \in L^q([-T,T];L^\rho)$ for any admissible exponent pair

(q,ρ), and the mass and energy conservation laws as well as the virial and Morawetz identities hold for u on the existence time interval $[-T,T]$: For any $t\in[-T,T]$,

$$M(u(t)) = M(u_0), \quad E(u(t)) = E(u_0),$$
$$\frac{1}{4}V(u(t)) - \frac{1}{4}V(u_0) = \int_0^t W(u(t'))dt',$$
$$\frac{1}{2}W(u(t)) - \frac{1}{2}W(u_0) = \int_0^t \int_{\mathbb{R}^d} |\nabla u|^2 dt + \kappa(\frac{d}{2} - \frac{d}{p+1}) \int_0^t \int_{\mathbb{R}^d} |u|^{p+1} dt.$$