Exercise 1

Recall the NonLinear Schrödinger equation
\[
\begin{cases}
  i\partial_t u + \Delta u = \kappa |u|^{p-1} u, \\
u|_{t=0} = u_0(x).
\end{cases}
\]
(NLS)

Let \(d = 1, 2, 1 + \frac{4}{d} \leq p < \infty\) and \(u_0 \in H^1(\mathbb{R}^d)\). Let \(u(t, x)\) be the solution of the Cauchy problem (NLS) satisfying \(\lim_{t \uparrow T^*} \|\nabla u(t)\|_{L^2(\mathbb{R}^d)} = +\infty\), then
\[
\|\nabla u(t)\|_{L^2(\mathbb{R}^d)} \geq C_0 (T^* - t)^{-\frac{1}{p-1}}, \quad \forall t \in [0, T^*).
\]

Hint: Recall the \(H^1\) local well-posedness results for \(d = 1, 2\). For any time \(t_0 < T^*\) with \(\|u(t_0)\|_{H^1(\mathbb{R}^d)} < \infty\), the solution \(u\) with \(\|u\|_{L^\infty([t_0, t_0+T]; H^1(\mathbb{R}^d))} \leq R := C\|u(t_0)\|_{H^1(\mathbb{R}^d)}\) exists at least on the time interval \([t_0, t_0+T]\), \(T > 0\) with
\[
T = C^{-1}\|u(t_0)\|_{H^1(\mathbb{R}^d)}^{\frac{q(1-p)}{q-p}},
\]
where \(q > p\).

Exercise 2

Recall the defocusing NonLinear Schrödinger equation
\[
\begin{cases}
  i\partial_t u + \Delta u = |u|^{p-1} u, \\
u|_{t=0} = u_0(x).
\end{cases}
\]
(NLS)

virial space
\[
\Sigma = \{ u \in H^1(\mathbb{R}^d) \mid xu \in L^2(\mathbb{R}^d) \}.
\]

Let \(1 + \frac{4}{d} \leq p < 2^* - 1\) and \(u_0 \in \Sigma\) and \(u \in C(\mathbb{R}; \Sigma)\) be the global-in-time solution of (NLS). Then
\[
u \in C(\mathbb{R}; \Sigma) \cap L^q(\mathbb{R}; W^{1,\rho}(\mathbb{R}^d)), \text{ with } (q, \rho) \text{ admissible exponent pair}
\]
Show that \(u\) scatters at large time in the sense that there exist two functions \(u_\pm \in \Sigma\) such that
\[
\lim_{t \to \pm\infty} \|u(t, \cdot) - S(t)u_\pm\|_\Sigma = 0.
\]

Hint: Recall in Theorem 3.2 we showed \(u\) scatters to \(u_\pm\) in \(L^2(\mathbb{R}^d)\) and \(H^1(\mathbb{R}^d)\). We define \(w(t, \cdot) = S(-t)u(t, \cdot)\) satisfies
\[
w(t) = u_0 - i \int_0^t S(-t')(|u|^{p-1} u)(t') dt'.
\]
Then we only need to show \(\|xw(t) - xu_+\|_{L^2(\mathbb{R}^d)} \to 0\) as \(t \to \infty\).

We follow the steps:
**Step 1:** We apply the operator \( P = x + 2it\nabla \) to the Duhamel formula

\[
u(t, x) = S(t)u_0 - i \int_0^t S(t-s)|u|^{p-1}u(s, \cdot)\, ds, \tag{Duhamel}\]

to get

\[
\|(x + 2it\nabla)u\|_{L^q([0, \infty); L^{p+1}(\mathbb{R}^d))} < +\infty.
\]

**Step 2:** We use the relation \( xS(-t) = S(-t)P(t) \) and Strichartz estimate to get

\[
\|(xw)(t) - (xw)(t')\|_{L^2(\mathbb{R}^d)} \to 0 \text{ as } t', t \to \infty.
\]

**Exercise 3**

Let \( 1 + \frac{4}{d} \leq p < 2^* - 1 \). Let \( u_+ \in \Sigma \). Show that the nonlinear map

\[
\Psi_+ : u \mapsto S(t)u_+ + i \int_t^\infty S(t-t')(|u|^{p-1}u)(t')\, dt',
\]
is well-defined and has a fixed point in the following complete metric space with \((q, \rho)\) admissible exponent pair:

\[
\tilde{X}_T = \{ u \in C([T, \infty); H^1(\mathbb{R}^d)) \mid Pu \in C([T, \infty); L^2(\mathbb{R}^d)), \}
\]

\[
\|u\|_{\tilde{X}_T} := \|u\|_{L^q([T, \infty); W^{1,\rho}(\mathbb{R}^d))} + \|Pu\|_{L^q([T, \infty); L^\rho(\mathbb{R}^d))} + \sup_{t \geq T} |t^{\frac{2}{q}}\|u(t)\|_{L^\rho(\mathbb{R}^d)} \leq R,
\]

for some appropriately chosen \( R \) and \( T \).

**Exercise 4**

Let \( d \geq 2 \). Recall the \( H^1_r(\mathbb{R}^d) \) space

\[
H^1_r(\mathbb{R}^d) = \{ f \in H^1(\mathbb{R}^d) \mid \exists \tilde{f} : [0, \infty) \to \mathbb{C} \text{ s.t. } f(x) = \tilde{f}(r), \quad r = \left( \sum_{j=1}^d |x_j|^2 \right)^{\frac{1}{2}} \}.
\]

Show that the embedding \( H^1_r(\mathbb{R}^d) \hookrightarrow L^2(\mathbb{R}^d) \) is not compact.

**Exercise 5**

Show that if \( u \in H^1(\mathbb{R}^d) \), then

\[
\int_{\mathbb{R}^d} |\nabla u|^2 \geq \int_{\mathbb{R}^d} |\nabla u|^2
\]

and if \( |u| > 0 \), then the above equality holds if and only if \( u = |u|e^{i\gamma} \) for some \( \gamma \in \mathbb{R} \).

**Hint:** We write \( u = f + ig, f, g \in H^1(\mathbb{R}^d; \mathbb{R}) \).

**Exercise 6**

Let \( t \in \mathbb{R}, h \in C^\infty_0(\mathbb{R}^d; \mathbb{R}) \) a radial, real-valued function and \( u \geq 0 \) a real-valued function . Show that

\[
\left| \int_{\mathbb{R}^d} \left( |u + th|^{p+1} - u^{p+1} - (p+1)thu^p \right) \, dx \right| \leq C \int_{\mathbb{R}^d} (t^2h^2u^{p-1} + th^p + 1|h|^{p+1}) \, dx.
\]
Exercise 7

Let \( \theta_\varepsilon = e^{\frac{|x|}{1+\varepsilon|x|}} \), \( \varepsilon > 0 \) be a bounded, Lipschitz continuous function with \( |\nabla \theta_\varepsilon|^2 \leq \theta_\varepsilon^2 \), a.e.

Let \( 1 < p < 2^* - 1 \) and \( v \) satisfies

\[
\Delta v - v + v^p = 0, \quad v \geq 0, \quad v \in H^1_r.
\]

Show that

\[
\int_{\mathbb{R}^d} e^{\frac{|x|}{1+\varepsilon|x|}} v^2 \, dx < \infty \quad (7.1)
\]

and

\[
\int_{\mathbb{R}^d} e^{\frac{|x|}{1+\varepsilon|x|}} |\nabla v|^2 \, dx < \infty. \quad (7.2)
\]

**Hint:** To show (7.1) we test the above equation by \( \theta_\varepsilon v \), then take \( \varepsilon \to 0 \).

To show (7.2) we apply \( \partial_{x_j} \) to the above equation and test it by \( \theta_\varepsilon \partial_{x_j} v \), then take \( \varepsilon \to 0 \).

Exercise 8

Let \( \rho_n : [0, \infty) \to [0, M] \) be positive monotone functions and there exists a continuous monotone function \( \rho(R) \) such that

\[
\forall R > 0, \quad \lim_{n \to \infty} \rho_n(R) = \rho(R).
\]

Let \( m = \lim_{R \to \infty} \rho(R) \leq M \).

Show that there exists a sequence \( R_n \to \infty \) such that

\[
m = \lim_{n \to \infty} \rho_n(R_n) = \lim_{n \to \infty} \rho_n\left(\frac{R_n}{2}\right) = \lim_{R \to \infty} \rho(R).
\]