Exercise 1

Recall the NonLinear Schrödinger equation

$$\begin{cases} i\partial_t u + \Delta u = \kappa |u|^{p-1}u, \quad \kappa = \pm 1, \\ u|_{t=0} = u_0(x). \end{cases} \tag{NLS}$$

Let $d = 1, 2$, $1 + \frac{4}{d} \leq p < \infty$ and $u_0 \in H^1(\mathbb{R}^d)$. Let $u(t,x)$ be the solution of the Cauchy problem (NLS) satisfying $\lim_{t\uparrow T^*} \|\nabla u(t)\|_{L^2(\mathbb{R}^d)} = +\infty$, then

$$\|\nabla u(t)\|_{L^2(\mathbb{R}^d)} \geq C_0(T^* - t)^{-\frac{1}{p-1}}, \quad \forall t \in [0, T^*).$$

Hint: Recall the H^1 local well-posedness results for $d = 1, 2$. For any time $t_0 < T^*$ with $\|u(t_0)\|_{H^1(\mathbb{R}^d)} < \infty$, the solution u with $\|u\|_{L^\infty([t_0,t_0+T];H^1(\mathbb{R}^d))} \leq R := C\|u(t_0)\|_{H^1(\mathbb{R}^d)}$ exists at least on the time interval $[t_0, t_0 + T]$, $T > 0$ with

$$T = C^{-1}\|u(t_0)\|_{H^1(\mathbb{R}^d)}^{\frac{q(1-p)}{q-p}},$$

where $q > p$.

Exercise 2

Recall the defocusing NonLinear Schrödinger equation

$$\begin{cases} i\partial_t u + \Delta u = |u|^{p-1}u, \\ u|_{t=0} = u_0(x). \end{cases} \tag{NLS}$$

virial space

$$\Sigma = \{u \in H^1(\mathbb{R}^d) \,|\, xu \in L^2(\mathbb{R}^d)\}.$$

Let $1 + \frac{4}{d} \leq p < 2^* - 1$ and $u_0 \in \Sigma$ and $u \in C(\mathbb{R}; \Sigma)$ be the global-in-time solution of (NLS). Then

$$u \in C(\mathbb{R}; \Sigma) \cap L^q(\mathbb{R}; W^{1,\rho}(\mathbb{R}^d)), \text{ with } (q, \rho) \text{ admissible exponent pair.}$$

Show that u scatters at large time in the sense that there exist two functions $u_\pm \in \Sigma$ such that

$$\lim_{t\to\pm\infty} \|u(t, \cdot) - S(t)u_\pm\|_\Sigma = 0.$$

Hint: Recall in Theorem 3.2 we showed u scatters to $u_\pm$ in $L^2(\mathbb{R}^d)$ and $H^1(\mathbb{R}^d)$. We define $w(t, \cdot) = S(-t)u(t, \cdot)$ satisfies

$$w(t) = u_0 - i\int_0^t S(-t'')(|u|^{p-1}u)(t'')\,dt''.$$

Then we only need to show $\|xw(t) - xu_+\|_{L_x^2(\mathbb{R}^d)} \to 0$ as $t \to \infty$.

We follow the steps:

Step 1: We apply the operator $P = x + 2it\nabla$ to the Duhamel formula

$$u(t,x) = S(t)u_0 - i\int_0^t S(t-s)|u|^{p-1}u(s,\cdot)\,ds, \qquad \text{(Duhamel)}$$

to get

$$\|(x+2it\nabla)u\|_{L^q([0,\infty);L^{p+1}(\mathbb{R}^d))} < +\infty.$$

Step 2: We use the relation $xS(-t) = S(-t)P(t)$ and Strichartz estimate to get

$$\|(xw)(t) - (xw)(t')\|_{L^2(\mathbb{R}^d)} \to 0 \text{ as } t', t \to \infty.$$

Exercise 3

Let $1 + \frac{4}{d} \leq p < 2^* - 1$. Let $u_+ \in \Sigma$. Show that the nonlinear map

$$\Psi_+ : u \mapsto S(t)u_+ + i\int_t^\infty S(t-t')(|u|^{p-1}u)(t')dt',$$

is well-defined and has a fixed point in the following complete metric space with (q,ρ) admissible exponent pair:

$$\tilde{X}_T = \{u \in C([T,\infty); H^1(\mathbb{R}^d)) \,|\, Pu \in C([T,\infty); L^2(\mathbb{R}^d)),$$
$$\|u\|_{\tilde{X}_T} := \|u\|_{L^q([T,\infty);W^{1,\rho}(\mathbb{R}^d))} + \|Pu\|_{L^q([T,\infty);L^\rho(\mathbb{R}^d))} + \sup_{t\geq T}|t|^{\frac{2}{q}}\|u(t)\|_{L^\rho(\mathbb{R}^d)} \leq R\},$$

for some appropriately chosen R and T.

Exercise 4

Let $d \geq 2$. Recall the $H^1_r(\mathbb{R}^d)$ space

$$H^1_r(\mathbb{R}^d) = \{f \in H^1(\mathbb{R}^d) \,|\, \exists \tilde{f} : [0,\infty) \to \mathbb{C} \text{ s.t. } f(x) = \tilde{f}(r),\ r = (\sum_{j=1}^d |x_j|^2)^{\frac{1}{2}}\}.$$

Show that the embedding $H^1_r(\mathbb{R}^d) \hookrightarrow L^2(\mathbb{R}^d)$ is not compact.

Exercise 5

Show that if $u \in H^1(\mathbb{R}^d)$, then

$$\int_{\mathbb{R}^d} |\nabla u|^2 \geq \int_{\mathbb{R}^d} |\nabla |u||^2$$

and if $|u| > 0$, then the above equality holds if and only if $u = |u|e^{i\gamma}$ for some $\gamma \in \mathbb{R}$.

Hint: We write $u = f + ig$, $f, g \in H^1(\mathbb{R}^d;\mathbb{R})$.

Exercise 6

Let $t \in \mathbb{R}$, $h \in C_0^\infty(\mathbb{R}^d;\mathbb{R})$ a radial, real-valued function and $u \geq 0$ a real-valued function . Show that

$$\Big|\int_{\mathbb{R}^d}\Big(|u+th|^{p+1} - u^{p+1} - (p+1)thu^p\Big)\,dx\Big| \leq C\int_{\mathbb{R}^d}\left(t^2h^2u^{p-1} + t^{p+1}|h|^{p+1}\right)\,dx.$$

Exercise 7

Let $\theta_\varepsilon = e^{\frac{|x|}{1+\varepsilon|x|}}$, $\varepsilon > 0$ be a bounded, Lipschitz continuous function with $|\nabla\theta_\varepsilon|^2 \le \theta_\varepsilon^2$, a.e.

Let $1 < p < 2^* - 1$ and v satisfies

$$\Delta v - v + v^p = 0, \quad v \ge 0, \quad v \in H_r^1.$$

Show that

$$\int_{\mathbb{R}^d} e^{|x|} v^2 \, dx < \infty \tag{7.1}$$

and

$$\int_{\mathbb{R}^d} e^{|x|} |\nabla v|^2 \, dx < \infty. \tag{7.2}$$

Hint: To show (7.1) we test the above equation by $\theta_\varepsilon v$, then take $\varepsilon \to 0$.

To show (7.2) we apply ∂_{x_j} to the above equation and test it by $\theta_\varepsilon \partial_{x_j} v$, then take $\varepsilon \to 0$.

Exercise 8

Let $\rho_n : [0,\infty) \to [0, M]$ be positive monotone functions and there exists a continuous monotone function $\rho(R)$ such that

$$\forall R > 0, \quad \lim_{n\to\infty} \rho_n(R) = \rho(R).$$

Let $m = \lim_{R\to\infty} \rho(R) \le M$.

Show that there exists a sequence $R_n \to \infty$ such that

$$m = \lim_{n\to\infty} \rho_n(R_n) = \lim_{n\to\infty} \rho_n(\frac{R_n}{2}) = \lim_{R\to\infty} \rho(R).$$