Exercise 1

Recall that, let $(u_n)_{n\geq 1}$ be a bounded sequence in $H^1(\mathbb{R}^d)$ with $\|u_n\|^2_{L^2(\mathbb{R}^d)} = M > 0$. There could be three situations:

- **Compactness:** There exists a sequence (y_n) in $\mathbb{R}^d$ such that
$$\forall q \in [2, 2^*), \quad u_n(\cdot - y_n) \to u \text{ in } L^q(\mathbb{R}^d) \text{ as } n \to \infty;$$
- **Evanescence:** $\forall q \in (2, 2^*)$, $u_n \to 0$ in $L^q(\mathbb{R}^d)$ as $n \to \infty$;
- **Dichotomy:** There exist two bounded sequences (v_n), (w_n) with compact supports in $H^1(\mathbb{R}^d)$ and $\alpha \in (0,1)$, such that
$$\begin{aligned}
&\operatorname{Supp} v_n \cap \operatorname{Supp} w_n = \{\}, \quad d(\operatorname{Supp} v_n, \operatorname{Supp} w_n) \to \infty \text{ as } n \to \infty,\\
&\|v_n\|^2_{L^2(\mathbb{R}^d)} \to \alpha M, \quad \|w_n\|^2_{L^2(\mathbb{R}^d)} \to (1-\alpha)M, \text{ as } n \to \infty,\\
&\forall q \in [2, 2^*), \quad \|u_n\|^q_{L^q} - \|v_n\|^q_{L^q} - \|w_n\|^q_{L^q} \to 0, \text{ as } n \to \infty,\\
&\liminf_{n\to\infty}(\|\nabla u_n\|^2_{L^2} - \|\nabla v_n\|^2_{L^2} - \|\nabla w_n\|^2_{L^2}) \geq 0.
\end{aligned}$$

Give three examples of bounded H^1 sequences such that compactness/evanescence/dichotomy hold respectively.

Exercise 2

Let $M > 0$, $1 < p < 1 + \frac{4}{d}$ i.e. $s_c = \frac{d}{2} - \frac{2}{p-1} < 0$ and the minimisation problem

$$J_M = \inf\{E(u) \,|\, u \in H^1(\mathbb{R}^d),\ \|u\|^2_{L^2(\mathbb{R}^d)} = M\}, \tag{1}$$

where $E(u) = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla u|^2 - \frac{1}{p+1}\int_{\mathbb{R}^d} |u|^{p+1}$.

If $u \geq 0$ is a minimiser of (1). Show that there exists $\tilde{\mu} = \tilde{\mu}(M) \in \mathbb{R}$ (independent of the minimisers) such that

$$\Delta u + u^p = \tilde{\mu} u, \quad u \geq 0, \quad u \in H^1(\mathbb{R}^d).$$

Hint: We follow the idea in the proof of Proposition 4.3 in the Lecture Notes.

Exercise 3

Let $u \geq 0$ and $u \in H^1(\mathbb{R}^d)$ be solution of the equation

$$\Delta u + u^p = \tilde{\mu} u. \tag{2}$$

Show that the following equalities hold:

$$\begin{aligned}
\frac{1}{2}\int_{\mathbb{R}^d} |\nabla u|^2 &= \frac{a}{p+1}\int_{\mathbb{R}^d} u^{p+1},\\
\tilde{\mu}\frac{1}{2}\int_{\mathbb{R}^d} |u|^2 &= \frac{b}{p+1}\int_{\mathbb{R}^d} u^{p+1},
\end{aligned}$$

where $a = \frac{d(p-1)}{4}, \quad b = \frac{d+2}{4} - \frac{(d-2)p}{4}$.

Hint: We test (2) by u and $(\frac{d}{2} + x \cdot \nabla)u$ respectively, then combine the results to get the desired equalities.

Exercise 4

Recall the NonLinear Schrödinger equation

$$\begin{cases} i\partial_t u + \Delta u = \kappa |u|^{p-1} u, \quad \kappa = \pm 1, \\ u|_{t=0} = u_0(x). \end{cases} \tag{NLS}$$

Recall the strong stability notion of the solitary waves $e^{it}Q(r)$ in H^1 for $1 < p < 1 + \frac{4}{d}$:
Let $u_0 \in H^1$ and let $u \in C(\mathbb{R}; H^1)$ be the global solution of (NLS) with the initial data u_0. Then for all $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that for all $u_0 \in H^1$:

$$\|u_0 - Q\|_{H^1} < \delta \text{ implies } \sup_{t \geq 0} \|u(t,x) - e^{it}Q(x)\|_{H^1} < \varepsilon.$$

Show that the strong stability property does not hold for (NLS) by the following examples:

- By scaling symmetry, for any $\lambda > 0$, there exists a solution $u_\lambda(t,x) = \lambda^{\frac{2}{p-1}} Q(\lambda x) e^{i\lambda^2 t}$ of (NLS) with the initial data $(u_0)_\lambda(x) = \lambda^{\frac{2}{p-1}} Q(\lambda x)$. Show that

  $$\|(u_0)_\lambda - Q\|_{H^1} \to 0 \text{ as } \lambda \to 1,$$

  while for any $\lambda \neq 1$,

  $$\sup_t \|u_\lambda(t,x) - e^{it}Q(x)\|_{H^1} \geq \|Q\|_{H^1}.$$

- By Galilean invariance, for any $v \in \mathbb{R}^d$, there exists a solution $u_v = e^{i(x \cdot v - |v|^2 t + t)} Q(x - 2vt)$ of (NLS) with the initial data $(u_0)_v = e^{iv \cdot x} Q(x)$. Show that

  $$\|(u_0)_v - Q\|_{H^1} \to 0 \text{ as } |v| \to 0,$$

  while

  whenever $v \neq 0$,

  $$\sup_t \|u_v(t,x) - e^{it}Q(x)\|_{H^1} \geq \|Q\|_{H^1}.$$