Exercise 1

Suppose that \( u(t, x) \in \mathcal{S}(\mathbb{R} \times \mathbb{R}^d) \) is a solution of the NonLinear Schrödinger equation:

\[
\begin{align*}
&i \partial_t u + \Delta u = \kappa |u|^{p-1} u, \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^d, \\
&u|_{t=0} = u_0(x),
\end{align*}
\]

where \( \kappa = \pm 1 \).

Show that we have the following conservation laws:

- **Mass conservation law**
  \[
  M(u)(t) = \int_{\mathbb{R}^d} |u(t, x)|^2 dx = M(u)(0),
  \]

- **Momentum conservation law**
  \[
  P_j(u)(t) = \Im \int_{\mathbb{R}^d} \bar{u} \partial_x^j u dx = P_j(u)(0), \quad 1 \leq j \leq d,
  \]

- **Energy conservation law**
  \[
  E(u)(t) = \frac{1}{2} \int_{\mathbb{R}^d} |\nabla u|^2 dx + \frac{\kappa}{p+1} \int_{\mathbb{R}^d} |u|^{p+1} dx = E(u)(0).
  \]

**Hint:** We use integration by part.

- Mass conservation law: test (NLS) by \( \bar{u} \), then take the imaginary part.
- Momentum conservation law: test (NLS) by \( \partial_x^j \bar{u} \), then take the real part.
- Energy conservation law: test (NLS) by \( \Delta \bar{u} - \kappa |u|^{p-1} \bar{u} \), then take the imaginary part.

Exercise 2

Consider the ODE system for the unknown vector-valued function \( j : \mathbb{R} \rightarrow \mathbb{C}^2 \), where \( x \in \mathbb{R} \) is the space variable

\[
\begin{pmatrix}
-iz \\
u
\end{pmatrix}
\begin{pmatrix}
j_x
\end{pmatrix} = \begin{pmatrix}
0 \\
uiz
\end{pmatrix} j.
\]

(2.1)

In the above \( z \in \mathbb{R} \) is a parameter, and \( u = u(x) \in L^1(\mathbb{R}; \mathbb{C}) \) is some known function.

Solve the boundary value problem for (2.1) with the following boundary condition

\[
j(x) = \begin{pmatrix}
e^{-ixz} \\
0
\end{pmatrix} + o(1) \quad \text{as} \quad x \to -\infty.
\]

**Hint:** We renormalise the solution \( l(x) = e^{ix} j(x) \), which satisfies the integral equation

\[
l(x) = \begin{pmatrix}
1 \\
0
\end{pmatrix} + \int_{-\infty}^{x} \begin{pmatrix}
0 \\
\bar{u}(x_1)
\end{pmatrix} l(x_1) dx_1.
\]

Then by iteration we show the existence and uniqueness of the solution.
Exercise 3

Consider the following overdetermined ODE systems for the unknown vector-valued function $\psi : \mathbb{R} \times \mathbb{R} \to \mathbb{C}^2$, where $(t,x)$ are time and space variables

$$\partial_x \psi = \begin{pmatrix} -iz & u \\ \overline{u} & iz \end{pmatrix} \psi := U(z,u)\psi,$$

$$\partial_t \psi = V(z,u)\psi.$$  \hspace{1cm} (3.1)

(3.2)

In the above $z$ is a parameter, and $u$ is some known function.

Search for $V$ such that the compatibility condition between (3.1) and (3.2) is an equation for $u$.

**Hint:** We make an Ansatz

$$V(z,u) = \sum_{j=0}^{k} z^{j-k} V_j(u),$$

and compare the coefficients of $z^{j-k}$ in the compatibility condition $\partial_x U = \partial_t V + [V,U]$. We can write $V_j = \begin{pmatrix} -ir_j & p_j \\ \overline{p}_j & ir_j \end{pmatrix}$, $V_0 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$, and search for the relation between $p_j$ and $r_j$, such that we can find countably many $V$. 

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