Winter term 2018/2019 **Karlsruher Institut für Technologie**
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Exercise sheet 1 M.Sc. Zihui He

Exercise 1

Suppose that $u(t,x) \in \mathcal{S}(\mathbb{R} \times \mathbb{R}^d)$ is a solution of the NonLinear Schrödinger equation:

$$\begin{cases} i\partial_t u + \Delta u = \kappa |u|^{p-1}u,\ t \in \mathbb{R},\ x \in \mathbb{R}^d, \\ u|_{t=0} = u_0(x), \end{cases} \tag{NLS}$$

where $\kappa = \pm 1$.

Show that we have the following conservation laws:

- Mass conservation law
$$M(u)(t) = \int_{\mathbb{R}^d} |u(t,x)|^2 \, dx = M(u)(0),$$
- Momentum conservation law
$$P_j(u)(t) = \mathfrak{Im} \int_{\mathbb{R}^d} \overline{u} \partial_{x_j} u \, dx = P_j(u)(0), \quad 1 \le j \le d,$$
- Energy conservation law
$$E(u)(t) = \frac{1}{2} \int_{\mathbb{R}^d} |\nabla u|^2 \, dx + \frac{\kappa}{p+1} \int_{\mathbb{R}^d} |u|^{p+1} \, dx = E(u)(0).$$

Hint: We use integration by part.

- Mass conservation law: test (NLS) by $\overline{u}$, then take the imaginary part.
- Momentum conservation law: test (NLS) by $\partial_{x_j}\overline{u}$, then take the real part.
- Energy conservation law: test (NLS) by $\Delta\overline{u} - \kappa|u|^{p-1}\overline{u}$, then take the imaginary part.

Exercise 2

Consider the ODE system for the unknown vector-valued function $j : \mathbb{R} \to \mathbb{C}^2$, where $x \in \mathbb{R}$ is the space variable

$$j_x = \begin{pmatrix} -iz & u \\ \overline{u} & iz \end{pmatrix} j. \tag{2.1}$$

In the above $z \in \mathbb{R}$ is a parameter, and $u = u(x) \in L^1(\mathbb{R}; \mathbb{C})$ is some known function.

Solve the boundary value problem for (2.1) with the following boundary condition

$$j(x) = \begin{pmatrix} e^{-izx} \\ 0 \end{pmatrix} + \mathrm{o}(1) \quad as \quad x \to -\infty.$$

Hint: We renormalise the solution $l(x) = e^{izx} j(x)$, which satisfies the integral equation

$$l(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_{-\infty}^{x} \begin{pmatrix} 0 & u(x_1) \\ e^{2iz(x-x_1)}\overline{u}(x_1) & 0 \end{pmatrix} l(x_1) \, dx_1.$$

Then by iteration we show the existence and uniqueness of the solution.

Exercise 3

Consider the following overdetermined ODE systems for the unknown vector-valued function $\psi : \mathbb{R} \times \mathbb{R} \to \mathbb{C}^2$, where (t, x) are time and space variables

$$\partial_x \psi = \begin{pmatrix} -iz & u \\ \overline{u} & iz \end{pmatrix} \psi := U(z, u)\psi, \tag{3.1}$$

$$\partial_t \psi = V(z, u)\psi. \tag{3.2}$$

In the above z is a parameter, and u is some known function.

Search for V such that the compatibility condition between (3.1) and (3.2) is an equation for u.

Hint: We make an Ansatz

$$V(z, u) = \sum_{j=0}^{k} z^{k-j} V_j(u),$$

and compare the coefficients of z^{k-j} in the compatibility condition $\partial_x U = \partial_t V + [V, U]$. We can write $V_j = \begin{pmatrix} -ir_j & p_j \\ \overline{p_j} & ir_j \end{pmatrix}$, $V_0 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$, and search for the relation between p_j and r_j, such that we can find countably many V.