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Notes for the lecture “Dispersive Equations” (Lecture 0170100, SS 2022, Weekly hours 3+1) [1]

[1] Comments are welcome to be sent to me by email.

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Main references:

- T. Cazenave: Semilinear Schrödinger equations. AMS, 2004.
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1 Introduction

In this lecture we will mainly consider the Cauchy problem for the semilinear Schrödinger equation

$$\begin{cases} i\partial_t u + \Delta_x u = \kappa |u|^{p-1} u, \\ u|_{t=0} = u_0(x). \end{cases} \tag{NLS}$$

Here $t \in \mathbb{R}$, $x \in \mathbb{R}^d$, $d \geqslant 1$ denote the time and space variables respectively, and $u = u(t,x) : \mathbb{R} \times \mathbb{R}^d \mapsto \mathbb{C}$ denotes the unknown wave function. The initial data $u_0 = u_0(x) : \mathbb{R}^d \mapsto \mathbb{C}$ is given.
The sign parameter κ takes value in $\{\pm 1\}$ and we call the nonlinear Schrödinger equation in (NLS) defocusing if $\kappa = 1$ (repulsive nonlinearity, which "cooperates" with Δu on LHS) and focusing if $\kappa = -1$ (attractive nonlinearity, which "competes" with Δu on LHS) respectively. The power parameter $p \in (1, \infty)$ is a real constant which indicates the "strongness" of the nonlinearity term, and if $p = 3$ we call (NLS) the cubic nonlinear Schrödinger equation.

Intuitively, we will investigate

- The wellposedness issue of this Cauchy problem (NLS). Roughly speaking, if $u_0 \in H^s(\mathbb{R}^d)$, does there exist a (unique) solution $u(t,x) \in C([-T,T]; H^s(\mathbb{R}^d))$ for some positive $T > 0$, and can T be arbitrarily large?
- The possible asymptotic behaviours of the solutions:
  - Blowup at finite time: $\lim_{t \to T_-} \|u(t,x)\|_{H^s_x(\mathbb{R}^d)} = \infty$ for some finite $T \in (0, \infty)$.

  - Scattering at infinite time: There exist two functions $u_{\pm}(x) \in H^s(\mathbb{R}^d)$ such that asymptotically $u(t,x) \sim e^{it\Delta_x} u_{\pm}$ as $t \to \pm\infty$.

- The stability of soliton solutions. For example, if $d = 1$, $\kappa = -1$, $p = 3$, then $\sqrt{2}e^{it}\operatorname{sech} x$ (with $\operatorname{sech}(x) = \frac{2}{e^x + e^{-x}}$ **Exercise: Check that it is a solution.**) is a solution of the cubic nonlinear Schrödinger equation, which decays exponentially fast at spacial infinity. Can we show the stability of such solutions in some appropriated sense?

We will see that the results will depend heavily on

- The space dimension d;
- The sign of κ;
- The nonlinearity exponent p;
- The functional space where the initial data u_0 stays in.

In this introduction we will first explain heuristically some basic concepts related to NLS:

- Dispersion;
- Semilinearity;
- Symmetries;
- Conservation laws;
- Solitons.

in Subsection 1.1. Then we will introduce some related models in Subsection 1.2.

1.1 Basic concepts

We are going to explain (heuristically) some basic concepts related to (NLS), such as dispersion, semilinearity, symmetries, conservation laws, solitary waves, in this subsection.

1.1.1 Dispersion

What does dispersion mean? Is the equation (NLS) dispersive? Roughly speaking, the dispersion means that "Waves with different frequencies travel at different velocities" and the dispersion property is related to the linear part of (NLS).

One dimensional case We now give some formal explanation. Let $d = 1$ and let $u(x,t)$ be a plane-wave solution

$$u(t,x) = e^{i(kx-\omega t)} = e^{ik(x-(\omega/k)t)}$$

of the linear Schrödinger equation

$$iu_t + u_{xx} = 0,$$

where k is the wave number (waves per unit length) and ω denotes the (angular) frequency. Then we derive the dispersion relation $\omega = \omega(k) = k^2$, such that

$$u(t,x) = e^{ik(x-kt)}$$

is a travelling wave with the phase velocity $c(k) = \omega(k)/k = k$ and the larger k is, the faster the wave travels, that is, high frequency waves travel much faster than low frequency waves! More generally, we take the inverse Fourier transform of the initial data

$$u_0(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ikx} \widehat{u_0}(k) dk,$$

then by superposition the solution of the linear Schrödinger equation reads (noticing that $e^{ik(x-c(k)t)}$ is the solution with initial data e^{ikx})

$$u(t,x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ik(x-c(k)t)} \widehat{u_0}(k) dk.$$

The fact that various Fourier modes travel at different speeds is considered to be dispersive phenomenon mathematically.
The well known Korteweg-de Vries (KdV) equation is also a nonlinear dispersive equation:

$$\partial_t u + u_{xxx} + uu_x = 0, \quad u|_{t=0} = u_0. \tag{KdV}$$

The linear part $u_t + u_{xxx} = 0$ has the phase velocity $c(k) = -k^2$.
There is an obvious example which is not a dispersive equation:

$$\partial_t u + c\partial_x u = 0, \quad u|_{t=0} = u_0, \text{ where } c = \textit{constant}.$$

The solution $u(t,x) = u_0(x-ct)$ travels at constant speed c.

General dimensional case Now we consider the general space dimension $d \geqslant 1$. We take the Fourier transform (for more details see my lecture notes "Fourier analysis and its applications to PDEs" in WS21/22) in x-variable to the Cauchy problem of the linear Schrödinger equation

$$i\partial_t u + \Delta_x u = 0, \quad u|_{t=0} = u_0,$$

to arrive at

$$\partial_t \widehat{u}(t,\xi) = -i|\xi|^2 \widehat{u}(t,\xi), \quad \widehat{u}(0,\xi) = \widehat{u_0}(\xi).$$

This is an ODE, and one can view ξ as a parameter, to derive the Fourier transform of the solution

$$\widehat{u}(t,\xi) = e^{-it|\xi|^2}\widehat{u_0}(\xi).$$

By inverse Fourier transform, we derive the solution

$$u(t,x) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{ix\cdot\xi} e^{-it|\xi|^2} \widehat{u_0}(\xi) d\xi = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{i\xi\cdot(x-t\xi)} \widehat{u_0}(\xi) d\xi,$$

and hence **(Exercise)**

$$u(t,x) = (4\pi i t)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{i\frac{|x-y|^2}{4t}} u_0(y) dy =: K_t * u_0, \tag{1.1}$$

where

$$K_t(x) = (4\pi i t)^{-\frac{d}{2}} e^{i\frac{|x|^2}{4t}}.$$

Notice that the kernel $K_t(x)$ does not decay at all in x-variable. Nevertheless we can already see the decay estimate in t-variable:

$$\|u(t,\cdot)\|_{L^\infty_x} \leqslant (4\pi|t|)^{-\frac{d}{2}} \|u_0\|_{L^1_x}. \tag{1.2}$$

Roughly speaking, all L^1-initial data will decay polynomially fast in L^∞-sense as the time goes to infinity.
One can compare the Cauchy problem for the linear Schrödinger equation with the Cauchy problem for the linear heat equation:

$$\partial_t u - \Delta_x u = 0, \quad u|_{t=0} = u_0,$$

where $t \geqslant 0$ and $x \in \mathbb{R}^d$. By Fourier transform, we derive **(Exercise)**

$$u(t,\cdot) = H_t * u_0, \quad t > 0, \tag{1.3}$$

where

$$H_t(x) = (4\pi t)^{-\frac{d}{2}} e^{-\frac{|x|^2}{4t}},$$

decays exponentially fast in x-variable for positive time $t > 0$. If $u_0 \in L^\infty$, then for all positive time $t > 0$, the solution $u(t,x)$ is smooth in x-variable, since $\partial_x^\alpha H_t(x)$ is integrable in x-variable for all indices α.

1.1.2 Semilinearity

The nonlinear Schrödinger equation (NLS) is of the semilinear form:

$$i\partial_t u + \Delta u = f(u), \text{ i.e. } \partial_t u = i\Delta u - if(u), \quad u|_{t=0} = u_0, \tag{1.4}$$

where the function f depends nonlinearly only on lower order terms: u (not on $\partial_t u, \nabla^2 u$)!
Recall the ODE theory. Consider the ODE of the form

$$v' = Lv + f(v), \quad v|_{t=0} = v_0,$$

where $v : \mathbb{R} \mapsto \mathbb{R}^d$ is the unknown, $t \in \mathbb{R}$ denotes the time variable, $L \in M_d(\mathbb{R})$ some linear transform and $f : \mathbb{R}^d \mapsto \mathbb{R}^d$ some function. We have the Duhamel formula for the solution

$$v(t) = e^{tL} v_0 + \int_0^t e^{(t-t')L} f(v(t'))\, \mathrm{d}t' .$$

Now we take the Fourier transform in x-variable to the equation (1.4)

$$\partial_t \widehat{u}(t,\xi) = -i|\xi|^2 \widehat{u}(t,\xi) - i\widehat{f(u)}(\xi), \quad \widehat{u}(0,\xi) = \widehat{u_0}(\xi).$$

View ξ as a parameter, then we also have (at least formally) the Duhamel formula for $\widehat{u}(\cdot;\xi)$

$$\widehat{u}(t,\xi) = e^{-it|\xi|^2}\widehat{u_0}(\xi) - i\int_0^t e^{-i(t-t')|\xi|^2} \widehat{f(u)}(t',\xi)\, \mathrm{d}t' .$$

By inverse Fourier transform, we derive the following Duhamel formula

$$u(t,x) = K_t * u_0 - i\int_0^t K_{t-t'} * f(u(t',\cdot))\, \mathrm{d}t' , \quad K_t = (4\pi i t)^{-\frac{d}{2}} e^{i\frac{|\cdot|^2}{4t}} . \tag{1.5}$$

1.1.3 Symmetries

We list here some interesting symmetries for the Schrödinger equation (NLS):

- Phase rotation symmetry: If $u(t,x)$ solves (NLS), then $e^{i\omega}u(t,x)$, $\omega \in \mathbb{R}$ also solves the Schrödinger equation in (NLS);
- Time/Space translation symmetry: If $u(t,x)$ solves (NLS), then $u_{t_0,x_0}(t,x) = u(t-t_0, x-x_0)$, $t_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}^d$ also solves the Schrödinger equation in (NLS);

- Space rotation symmetry: If $u(t,x)$ solves (NLS), then $u(t,\Omega x)$, $\Omega \in SO(d)$ also solves the Schrödinger equation in (NLS);
- Time reversal symmetry: If $u(t,x)$ solves (NLS), then $\bar{u}(-t,x)$ ($\bar{u}$ means the complex conjugate of u) also solves the Schrödinger equation in (NLS);
- Galilean invariance: If $u(t,x)$ solves (NLS), then $e^{i(x\cdot v-|v|^2t)}u(t,x-2vt)$, $v \in \mathbb{R}^d$ also solves the Schrödinger equation in (NLS);
- Pseudo-conformal symmetry for the mass critical case $p = 1 + \frac{4}{d}$: If $u(t,x)$ solves (NLS), then $\frac{e^{i\frac{|x|^2}{4t}}}{t^{\frac{d}{2}}}u(-\frac{1}{t},\frac{x}{t})$, $\frac{e^{i\frac{|x|^2}{4(1+t)}}}{(1+t)^{\frac{d}{2}}}u(\frac{t}{1+t},\frac{x}{1+t})$, etc. $t > 0$ also solve the Schrödinger equation in (NLS);
- Scaling symmetry: If $u(t,x)$ solves (NLS), then $u_\lambda(t,x) = \frac{1}{\lambda^{\frac{2}{p-1}}}u(\frac{t}{\lambda^2},\frac{x}{\lambda})$, $0 < \lambda \in \mathbb{R}$ also solves the Schrödinger equation in (NLS).

Scaling symmetry and sub-/super-critical cases Let us focus on the scaling symmetry for a while: Notice that the scaling symmetry includes both linear and nonlinear informations in the nonlinear Schrödinger equation (NLS). Recall that the Sobolev space $H^s(\mathbb{R}^d)$, $s \in \mathbb{R}$ is defined to be

$$H^s(\mathbb{R}^d) := \{f \in \mathcal{S}'(\mathbb{R}^d) \mid \|f\|^2_{H^s(\mathbb{R}^d)} := \int_{\mathbb{R}^d}(1+|\xi|^2)^s|\hat{f}(\xi)|^2\,\mathrm{d}\xi < \infty\}, \tag{1.6}$$

and we also define the homogeneous Sobolev seminorm as

$$\|f\|^2_{\dot{H}^s(\mathbb{R}^d)} := \int_{\mathbb{R}^d}|\xi|^{2s}|\widehat{f}(\xi)|^2 d\xi. \tag{1.7}$$

Denote the critical exponent

$$s_c = \frac{d}{2} - \frac{2}{p-1}. \tag{1.8}$$

Let the initial data $u_0 \in H^s(\mathbb{R}^d)$, then the rescaled initial datum $u_{0,\lambda}(x) = \frac{1}{\lambda^{\frac{2}{p-1}}}u_0(\frac{x}{\lambda})$, $\lambda > 0$ has $\dot{H}^s(\mathbb{R}^d)$-norm as follows **(Exercise)**

$$\|u_{0,\lambda}\|_{\dot{H}^s(\mathbb{R}^d)} = \lambda^{-s+s_c}\|u_0\|_{\dot{H}^s(\mathbb{R}^d)}.$$

Heuristically, we then divide the regularity exponent s of the Sobolev space H^s into three cases:

- $s > s_c$ (subcritical case)
  If the solution $u(t,x)$ with the initial data of size 1: $\|u_0\|_{\dot{H}^s} = 1$ exists on the time interval $[0, T_*]$, then as $\lambda \to \infty$, the rescaled initial data $\|u_{0,\lambda}\|_{\dot{H}^s(\mathbb{R}^d)} \to 0$ and the rescaled solution u_λ exists on the time interval $[0, \lambda^2 T_*]$ with $\lambda^2 T_* \to \infty$.

  This is the most favourable situation in well-posedness issue: we can make both the small initial norm and the long time interval at the same time.

- $s = s_c$ (critical case)
  It is easy to see that the $\dot{H}^{s_c}$-norm is invariant under the scaling: $\|u_{0,\lambda}\|_{\dot{H}^{s_c}(\mathbb{R}^d)} = \|u_0\|_{\dot{H}^{s_c}(\mathbb{R}^d)}$, and as $\lambda \to \infty$ the rescaled existing time interval is still $[0, \lambda^2 T_*]$ with $\lambda^2 T_* \to \infty$. This is a unclear situation for the wellposedness result.

- $s < s_c$ (supercritical case)
  In this case as $\lambda \to \infty$, $\|u_{0,\lambda}\|_{\dot{H}^s(\mathbb{R}^d)} \to \infty$ as $\lambda^2 T_* \to \infty$, that is, the growing norm corresponds to longer time interval. Blowup may happen in this situation.

In particular, we are in the L^2 (mass)-subcritical/critical/supercritical case if

$$0 = s > / = / < s_c, \text{ i.e. } p < / = / > 1 + \frac{4}{d},$$

and we are in the H^1 (energy)-subcritical/critical/supercritical case if

$$1 = s > / = / < s_c, \text{ i.e. } p < / = / > 1 + \frac{4}{d-2}.$$

In this lecture we take the convention that $1 + \frac{4}{d-2} = \infty$ if $d = 1, 2$. It seems that we should have well-posedness results in L^2 or H^1 framework when $p < 1 + \frac{4}{d}$ or $p < 1 + \frac{4}{d-2}$ and we will indeed prove this in Section 2.

1.1.4 Conservation laws

Suppose that $u(t,x)$ is a smooth and fast decaying solution of the Schrödinger equation in (NLS). We have the following conservation laws a priori **(Exercise)**:

- Mass conservation law

$$M(u)(t) = \int_{\mathbb{R}^d} |u(t,x)|^2 \, \mathrm{d}x = M(u)(0). \tag{1.9}$$

  Indeed, we test this Schrödinger equation by $\bar{u}$ and then take the imaginary part to obtain (1.9).

- Momentum conservation law

$$P_j(u)(t) = \mathrm{Im} \int_{\mathbb{R}^d} \bar{u} \partial_{x_j} u \, \mathrm{d}x \; = P_j(u)(0). \tag{1.10}$$

Indeed, we test the Schrödinger equation by $\partial_{x_j} \bar{u}$ and then take the real part to arrive at (1.10).

- Energy conservation law

$$E(u)(t) = \frac{1}{2} \int_{\mathbb{R}^d} |\nabla u|^2 \, \mathrm{d}x \; + \frac{\kappa}{p+1} \int_{\mathbb{R}^d} |u|^{p+1} \, \mathrm{d}x \; = E(u)(0). \tag{1.11}$$

Indeed, we test the Schrödinger equation by $\Delta \bar{u} - \kappa |u|^{p-1} \bar{u}$ and then take the imaginary part to get (1.11).

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Remark 1.1. *These above conservation laws obviously hold true for smooth and fast decaying solutions for* (NLS). *Nevertheless they can also hold for less regular solutions by the approximation argument, e.g. the mass conservation law* (1.9) *will hold for L^2-subcritical case with L^2 initial data. These conservation laws will help us to get global-in-time well-posedness result, see Subsection 2.2.2 below.*

We are going to revisit these conservation laws from other interesting viewpoints (intuitively, but not rigorously).

1. Lagrangian viewpoint Recall in the classical Lagrangian mechanics that if $L = L(q, \dot{q}) : TM \mapsto \mathbb{R}$ (M is some n-dimensional manifold and TM is its tangential bundle, e.g. $M = \mathbb{R}^n$ and $TM = \mathbb{R}^n \times \mathbb{R}^n$) denotes some Lagrangian, which admits a one parameter group of functions $\tau_\theta : M \mapsto M$, that is,

$$L((\tau_\theta)_* v) = L(v), \quad \forall v \in TM,$$

then Noether Theorem ensures a conservation law $I : TM \mapsto \mathbb{R}$, which (in local coordinates) reads as

$$I(q, \dot{q}) = \frac{\partial L}{\partial \dot{q}} \frac{d}{d\theta}\Big|_{\theta=0} \tau_\theta(q).$$

We would like to adopt this Lagrangian viewpoint to relate the phase rotation/time translation/space translation symmetries to the mass/energy/momentum conservation laws. We will mainly follow Chapter 8 in Ginibre's lecture notes "Introduction aux équations de Schrödinger non linéaires".

Abstract setting For notational simplicity we denote $y = (t, x)$ as a point in the time-space $\mathbb{R}^{1+d}$, where

$$y_0 = t, \quad y_j = x_j, \quad j = 1, \cdots, d, \quad \partial_\mu = \frac{\partial}{\partial y_\mu}, \quad \mu = 0, 1, \cdots, d.$$

We consider a Lagrangian function

$$L : \mathbb{C} \times \mathbb{C}^{1+d} \mapsto \mathbb{R},$$

and for given functions $u, \partial_\mu u : \mathbb{R}^{1+d} \mapsto \mathbb{C}$, $\mu = 0, 1, \cdots, d$, (with the canonical coordinates $u, \bar{u}, \partial_\mu u, \partial_\mu \bar{u}$), with an abuse of notation we denote

$$L(y) = L(u(y), (\partial_\mu u)(y)).$$

Step 1. Definitions of δu, $\delta \partial_\mu u$, δL. [2] Let y be fixed for the moment, and let $\{\tau_\theta\}_{\theta \in \mathbb{R}}$ be a one parameter group of transformations $u \mapsto \tau_\theta u$, and we define the associated variation $\delta : u \mapsto \delta u$ as

$$\delta u(y) = \frac{d}{d\theta}\Big|_{\theta=0} (\tau_\theta u)(y).$$

Correspondingly for $\partial_\mu u \mapsto \tau_\theta \partial_\mu u = \partial_\mu \tau_\theta u$,

$$\delta \partial_\mu u(y) = \frac{d}{d\theta}\Big|_{\theta=0} (\tau_\theta \partial_\mu u)(y) = \partial_\mu \delta u(y).$$

Then we define correspondingly

$$(\tau_\theta L)(y) = L(\tau_\theta u(y), (\tau_\theta \partial_\mu u)(y)),$$

and hence

$$\begin{aligned}\delta L(y) = \frac{d}{d\theta}\Big|_{\theta=0} (\tau_\theta L)(y) &= \frac{\partial L}{\partial u} \delta u(y) + \sum_{\mu=0}^{d} \frac{\partial L}{\partial(\partial_\mu u)} \delta \partial_\mu u(y) \\ &= \sum_{\mu=0}^{d} \partial_\mu \Big(\frac{\partial L}{\partial(\partial_\mu u)} \delta u(y) \Big) + \Big(\frac{\partial L}{\partial u} - \sum_{\mu=0}^{d} \partial_\mu \frac{\partial L}{\partial(\partial_\mu u)} \Big) \delta u(y).\end{aligned}$$

Step 2. Definition of δy. Now with an abuse of notations, let $\{\tau_\theta\}_{\theta \in \mathbb{R}}$ be a group of transformation $y \mapsto \tau_\theta y$, and we define

$$\delta y = \frac{d}{d\theta}\Big|_{\theta=0} (\tau_\theta y).$$

[2] There are other ways to define the variations. The definitions below are not compatible with the classical definitions where $q = q(t)$, and δ and $\frac{d}{dt}$ can not commute.

For example, for $a \in \mathbb{R}^{1+d}$,

$$
\begin{aligned}
&\text{If } \tau_\theta u(y) = u(y - \theta a), \text{ then } \delta u(y) = -\sum_{\lambda=0}^{d} a_\lambda \partial_\lambda u(y), \\
&\text{If } \tau_\theta y = y + \theta a, \text{ then } \delta y = a, \text{ and we have } (\tau_\theta u)(\tau_\theta y) = u(y).
\end{aligned} \tag{TL}
$$

Step 3. Definition of δA_Λ. We now turn to the consideration of the action

$$A_\Lambda = \int_\Lambda L(y) dy,$$

with respect to some Lipschitz-domain $\Lambda \subset \mathbb{R}^{1+d}$, and define

$$
\begin{aligned}
\tau_\theta A_\Lambda &= \int_{\tau_\theta \Lambda} (\tau_\theta L)(y) dy, \\
\delta A_\Lambda &= \frac{d}{d\theta}\Big|_{\theta=0} (\tau_\theta A_\Lambda) = \int_{\partial\Lambda} L(y)(\delta y \cdot n) d\sigma + \int_\Lambda \delta L(y) dy,
\end{aligned}
$$

then recalling the definition of δL, and we arrive at by integration by parts

$$\delta A_\Lambda = \int_{\partial\Lambda} \sum_{\mu=0}^{d} \Big(\frac{\partial L}{\partial(\partial_\mu u)} \delta u + L(y) \delta y_\mu \Big) n_\mu d\sigma + \int_\Lambda \Big(\frac{\partial L}{\partial u} - \sum_{\mu=0}^{d} \partial_\mu \frac{\partial L}{\partial(\partial_\mu u)} \Big) \delta u(y) dy.$$

On the other side, we can do change of variables to arrive at

$$
\begin{aligned}
\tau_\theta A_\Lambda &= \int_{\tau_\theta \Lambda} (\tau_\theta L)(\tau_\theta y) d(\tau_\theta y) \\
&= \int_\Lambda (\tau_\theta L)(\tau_\theta y) j(\theta, y) dy,
\end{aligned}
$$

where $j(\theta, y)$ is the Jacobian of the transformation $y \mapsto \tau_\theta y$. Hence if

$$(\tau_\theta L)(\tau_\theta y) j(\theta, y) = L(y) \tag{TF}$$

holds, then we have $\delta A_\Lambda = 0$.

Step 4. Euler-Lagrangian equation. The variational problem is to search for the critical points u such that the action is stationary: $\delta A_\Lambda = 0$, with respect to the variation of u which keeps at the same time the boundary value $u|_{\partial\Lambda}$ invariant.

The fact that $\delta A_\Lambda = 0$ vanishes for all Λ and for all δu which vanishes on the boundary $\partial\Lambda$, is equivalent to the fact that the critical points u satisfies the Euler-Lagrangian equation

$$\frac{\partial L}{\partial u} - \sum_{\mu=0}^{d} \partial_\mu \frac{\partial L}{\partial(\partial_\mu u)} = 0. \tag{EL}$$

Step 5. Conservation laws. If we take $\Lambda = [t_1, t_2] \times \mathbb{R}^d$ for arbitrarily $t_1, t_2 \in \mathbb{R}$, then for the solutions u of (EL) it holds

$$\delta A_\Lambda = \int_{\mathbb{R}^d} \Big(\frac{\partial L}{\partial(\partial_0 u)} \delta u + L \delta y_0 \Big) dx \Big|_{t_1}^{t_2}.$$

Hence if (TF) holds, then we arrive at the conservation laws for the solutions u of the Euler-Lagrangian equation (EL):

$$Q(u)(t) = \int_{\mathbb{R}^d} \Big(\frac{\partial L}{\partial(\partial_0 u)} \delta u + L \delta y_0 \Big) dx. \tag{CL}$$

NLS case We take the Lagrangian $L : \mathbb{C} \times \mathbb{C}^{d+1} \mapsto \mathbb{R}$ defined as follows:

$$L_{\text{NLS}}(u, u_0, u_1, \cdots, u_d) = \frac{i}{2}(\bar{u} u_0 - u \overline{u_0}) - |(u_1, \cdots, u_d)|^2 - \frac{2\kappa}{p+1} |u|^{p+1}. \tag{1.12}$$

Then the nonlinear Schrödinger equation (NLS) can be viewed as the Euler-Lagrangian equation (EL) associated to the Lagrangian L_{NLS} **(Exercise)**. Now we take u as the solution of (NLS), then for the transformations satisfying (TF), we have the conservation law (CL):

$$Q(u)(t) = \int_{\mathbb{R}^d} \Big(\frac{i}{2} \bar{u} \delta u - \frac{i}{2} u \delta \bar{u} + L_{\text{NLS}} \delta y_0 \Big) dx.$$

Recall the symmtries in Subsection 1.1.3, and we will derive the corresponding conservation laws:

- Phase rotation symmetry. Let

  $$\tau_\theta u = e^{i\theta} u, \quad \tau_\theta y = y,$$

  such that $\delta y = 0$, $\delta u = iu$, $\delta \bar{u} = -i\bar{u}$. Then (TF) holds, and the conservation law (CL) reads as the mass conservation law:

  $$Q(u)(t) = -\int_{\mathbb{R}^d} |u|^2 \, \mathrm{d}x = -M(u)(t).$$

- Time-space translation. Let $a = (t_0, x_0) \in \mathbb{R}^{1+d}$, and we take the time-space transformation group (TL), such that (TF) holds. Then the conservation law (CL) read as momentum conservation law (with respect to space translation) and energy conservation law (with respect to time translation) respectively

  $$Q(u)(t) = -2E(u) t_0 + \sum_{j=1}^{d} P_j(u) \cdot x_{0,j}.$$

Interested readers may try other symmetries.

2. Complete integrability case We consider the following one-dimensional defocusing cubic nonlinear Schrödinger equation (i.e. we take $d = 1$, $\kappa = 1$, $p = 3$ and scaling $u \mapsto \frac{1}{\sqrt{2}}u$ in (NLS))

$$i\partial_t u + u_{xx} = 2|u|^2 u, \quad u|_{t=0} = u_0. \tag{1.13}$$

Lax pair reformulation By [Zakharov-Shabat 1972], the equation (1.13) can be viewed **(Exercise)** as the compatibility condition of the two systems

$$\psi_x = \begin{pmatrix} -iz & u \\ \bar{u} & iz \end{pmatrix} \psi =: U\psi, \tag{1.14}$$

$$\psi_t = i \begin{pmatrix} -2z^2 - |u|^2 & -2izu + u_x \\ -2iz\bar{u} - \bar{u}_x & 2z^2 + |u|^2 \end{pmatrix} \psi =: V\psi, \tag{1.15}$$

where in these two systems z is a parameter, (t, x) are space and time variables, u is some given function and $\psi : \mathbb{R} \times \mathbb{R} \to \mathbb{C}^2$ is the unknown vector-valued function. Here the compatibility condition of the above two systems means that

$$\begin{aligned} \psi_{xt} &= (U\psi)_t = U_t\psi + U\psi_t = U_t\psi + UV\psi \\ \text{and } \psi_{tx} &= (V\psi)_x = V_x\psi + V\psi_x = V_x\psi + VU\psi \end{aligned}$$

should be the same, that is,

$$U_t = V_x + [V, U] \text{ with } [V, U] := VU - UV. \tag{1.16}$$

We correspondingly have the Lax-pair formulation of cubic NLS (1.13). We rewrite the system (1.14) into the form of the spectral problem of the self-adjoint Lax operator L (with the domain depending on the potential u, e.g. $D(L) = H^1(\mathbb{R}) \subset L^2(\mathbb{R})$ if $u \in L^\infty(\mathbb{R})$) as follows

$$L\psi = z\psi, \quad L = \begin{pmatrix} i\partial_x & -iu \\ i\bar{u} & -i\partial_x \end{pmatrix}. \tag{1.17}$$

Then we can replace $z\psi$ by $L\psi$ in the system (1.15) to get

$$\begin{aligned} \psi_t = P\psi, \quad P &= 2i \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} L^2 + 2 \begin{pmatrix} 0 & u \\ \bar{u} & 0 \end{pmatrix} L + i \begin{pmatrix} -|u|^2 & u_x \\ -\bar{u}_x & |u|^2 \end{pmatrix} \\ &= i \begin{pmatrix} 2\partial_x^2 - |u|^2 & -u\partial_x - \partial_x u \\ \bar{u}\partial_x + \partial_x \bar{u} & -2\partial_x^2 + |u|^2 \end{pmatrix}. \end{aligned} \tag{1.18}$$

The compatibility condition (1.16), i.e. the cubic nonlinear Schrödinger equation (1.13) equals to the following evolutionary equation

$$L_t = [P, L], \tag{1.19}$$

and the operator pair (L, P) is called Lax pair for (1.13).

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Transmission/Reflection coefficients Formally, thanks to the evolutionary equation (1.19), if $L\psi = z\psi$ then z is independent of the time since we derive $z_t = 0$ from $(L\psi)_t = (z\psi)_t$:

$$\begin{aligned}(L\psi)_t &= L_t\psi + L\psi_t = [P, L]\psi + LP\psi = PL\psi,\\ (z\psi)_t &= z_t\psi + z\psi_t = z_t\psi + zP\psi = z_t\psi + PL\psi.\end{aligned}$$

This fact is non trivial since L depends on $u(t, x)$ and hence its spectrum should depend on t generally. More precisely we will indeed consider $z \in \mathbb{R}$ (the continuous spectrum of the self-adjoint operator L) and define

- Transmission coefficient $T(z)$
- Reflection coefficient $R(z)$

associated to the Lax operator L. Roughly speaking, if there is an incident wave $\begin{pmatrix} e^{-izx} \\ 0 \end{pmatrix}$ coming from the right $+\infty$, then by the "scattering" of L, it will be transmissed to the left $-\infty$ as the outcoming wave $T(z)\begin{pmatrix} e^{-izx} \\ 0 \end{pmatrix}$ with the complex amplitude $T(z)$, and be reflected to the right as the outcoming wave $R(z)\begin{pmatrix} 0 \\ e^{izx} \end{pmatrix}$ with the complex amplitude $R(z)$.

More precisely, there are two fundamental solution matrices $J^{\pm}$ of the ordinary differential equation (1.14), where $J^{\pm}$ satisfy the following "initial data" at infinity (we suppose that u vanishes at infinity)

$$\lim_{x\to\pm\infty} (J^{\pm}(x)e^{izx\sigma_3} - \mathrm{Id}\,) = 0, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

By the ODE theory, there exists an invertible matrix $S = S(z)$ (independent of x) such that

$$J^-(x; z) = J^+(x; z)S(z)^{-1},$$

where the scattering matrix $S(z)$, $z \in \mathbb{R}$ should **(Exercise)**

- be of the form $S(z) = \begin{pmatrix} \bar a & -\bar b \\ -b & a \end{pmatrix}$, by virtue of the symmetry of the ODE (1.14): $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \mapsto \begin{pmatrix} \overline{\psi_2} \\ \overline{\psi_1} \end{pmatrix}$, such that $\overline{J^\pm} = \sigma_1 J^\pm \sigma_1$ and hence $\overline{S} = \sigma_1 S \sigma_1$, where $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$;
- with $\det(S) = |a|^2 - |b|^2 = 1$, by virtue of the fact that $\det(J^\pm(x;z)) = \lim_{x\to\pm\infty} \det(J^\pm(x;z)) = 1$ are independent of the space variable x. Hence $|a| \geqslant 1$ never vanishes.

and hence $S^{-1} = \begin{pmatrix} a & \bar b \\ b & \bar a \end{pmatrix}$. Or in column vector formulation,

$$(j^{-,1}\ j^{-,2}) = (j^{+,1}\ j^{+,2}) \begin{pmatrix} a & \bar b \\ b & \bar a \end{pmatrix},$$

and hence $j^{-,1} = aj^{+,1} + bj^{+,2}$, that is,

$$a^{-1} j^{-,1} = j^{+,1} + (b/a) j^{+,2},$$

where $j^{\pm,1}(x) \sim \begin{pmatrix} e^{-izx} \\ 0 \end{pmatrix}$ as $x \to \pm\infty$, and $j^{\pm,2}(x) \sim \begin{pmatrix} 0 \\ e^{izx} \end{pmatrix}$ as $x \to \pm\infty$. Thus $T = a^{-1}$ and $R = b/a$. If $u = 0$, then $a = 1$, $b = 0$.

Solve (1.14) *by Picard iteration.* More precisely, we solve (1.14) with the initial data $j^{-,1}(x) \sim \begin{pmatrix} e^{-izx} \\ 0 \end{pmatrix}$, $x \to -\infty$ by Picard iteration: We consider indeed $l := e^{izx} j^{-,1}$, which solves

$$l_x = \begin{pmatrix} 0 & u \\ \bar u & 2iz \end{pmatrix} l, \quad \lim_{x\to-\infty} l(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Then we get the solution by iteration

$$l = \sum_{n=0}^\infty l_n, \quad l_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$
$$l_{n+1}(x) = \int_{x_1<x} e^{2iz(x-x_1)} \begin{pmatrix} 0 & u(x_1) \\ \bar u(x_1) & 0 \end{pmatrix} l_n(x_1)\, \mathrm{d}x_1.$$

We write the first component of the solution as

$$l^1 = 1 + \int_{x_1<y_1<x} e^{2iz(y_1-x_1)} u(y_1) \bar u(x_1)\, \mathrm{d}x_1\, \mathrm{d}y_1 + \cdots$$

$$+\int_{x_1<y_1<\cdots<x_n<y_n<x} e^{2iz(y_1-x_1+\cdots+y_n-x_n)}u(y_1)\bar{u}(x_1)\cdots u(y_n)\bar{u}(x_n)\,\mathrm{d}x_1\cdots\mathrm{d}x_n\,\mathrm{d}y_1\cdots\mathrm{d}y_n+\cdots$$

Thus $a = T^{-1}$ is the limit $\lim_{x\to\infty} l^1(x)$:

$$\begin{aligned}a = T^{-1}(z) = 1 + &\int_{x_1<y_1} e^{2iz(y_1-x_1)}u(y_1)\bar{u}(x_1)\,\mathrm{d}x_1\,\mathrm{d}y_1+\cdots\\ &+\int_{x_1<y_1<\cdots<x_n<y_n} e^{2iz(y_1-x_1+\cdots+y_n-x_n)}u(y_1)\bar{u}(x_1)\cdots u(y_n)\bar{u}(x_n)\,\mathrm{d}x_1\cdots\mathrm{d}x_n\,\mathrm{d}y_1\cdots\mathrm{d}y_n+\cdots\end{aligned} \tag{1.20}$$

We observe that if $u \in L^1(\mathbb{R};\mathbb{C})$, then for $z \in \mathbb{R}$, the above iteration procedure is well-defined, and we have a unique solution l. In particular, with $m_k = \int_{-\infty}^{x_k} |u|$ such that $dm_k = |u(x_k)|dx_k$ and $m = \|u\|_{L^1}$,

$$\begin{aligned}|T^{-1}(z)| &\leqslant 1 + \int_0^m\int_0^{m_2} dm_1dm_2 + \int_0^m\int_0^{m_4}\int_0^{m_3}\int_0^{m_2} dm_1dm_2dm_3dm_4+\cdots\\ &\leqslant \sum_{n=0}^{\infty}\frac{1}{(2n)!}\|u\|_{L^1}^{2n} \leqslant e^{\|u\|_{L^1}^2}.\end{aligned}$$

Solve the compatible two ODE systems (1.14)-(1.15). We can show that $W^{\pm}(t,x;z) = J^{\pm}(t,x;z)e^{-2iz^2t\sigma_3}$ are simultaneous fundamental solutions of the compatible linear systems (1.14)-(1.15). Then the evolution of the scattering matrix is described by $a(t;z) = a(0;z)$ and $b(t;z) = e^{4iz^2t}b(0;z)$. **(Exercise)**

We have indeed

- $T(z)$ does not depend on time and hence gives infinitely many conservation laws for the cubic NLS (1.13). If e.g. $u(x) \in L^1$, we can indeed extend the definition of the transmission coefficient to the upper half complex plane (if $\operatorname{Im} z > 0$, then $e^{2izx} = e^{-2(\operatorname{Im} z)x}e^{2i(\operatorname{Re} z)x}$ has exponential decay as $x \to \infty$) and have the following asymptotic expansion at infinity:

  $$\ln T^{-1}(z) \sim i\sum_{k=0}^{\infty} H_k(2z)^{-(k+1)}, \quad \operatorname{Im} z > 0, \quad |z| \to \infty.$$

  The first three conservation lawsare the conserved mass, momentum, energy defined in (1.9)-(1.10)-(1.11) respectively:

  $$H_0 = M, \quad H_1 = -P, \quad H_2 = E.$$

By the conservation of $T(z)$, all the coefficients H_k in the above expansion are conserved by the cubic NLS flow and hence we derived infinitely many conservation laws.

In particular, if $\|u\|_{L^1} \ll 1$ is sufficiently small, then the leading term in $\ln T^{-1}(z)$ reads as (recalling (1.20))

$$T_2(z) := \int_{x_1<y_1} e^{2iz(y_1-x_1)} u(y_1)\overline{u(x_1)}\, dx_1\, dy_1.$$

By use of (inverse) Fourier transform, one can rewrite it as **(Exercise)**

$$T_2(z) = \int_{\mathbb{R}} \frac{i}{2z+\xi}\, \widehat{u}(\xi)\, \overline{\widehat{u}(\xi)}\, d\xi.$$

If $2z = i\tau$ for some $\tau \in \mathbb{R}^+$, then

$$T_2(\frac{i}{2}\tau) = \int_{\mathbb{R}} \frac{\tau + i\xi}{\xi^2+\tau^2} |\hat{u}(\xi)|^2\, \mathrm{d}\xi\,,$$

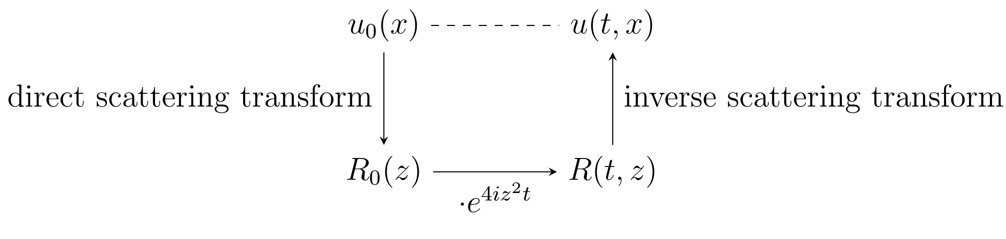
which can be expanded asymptotically as

$$\sum_{l=0}^{\infty} (-1)^l \tilde{H}_{2l} \tau^{-(2l+1)} + i \sum_{l=0}^{\infty} (-1)^l \tilde{H}_{2l+1} \tau^{-(2l+2)}, \quad \tau \to \infty,$$

where

$$\begin{aligned} \tilde{H}_{2l} &= \int_{\mathbb{R}} \xi^{2l} |\hat{u}(\xi)|^2\, \mathrm{d}\xi \ = \|u\|_{\dot{H}^l}^2, \\ \tilde{H}_{2l+1} &= \int_{\mathbb{R}} \xi^{2l+1} |\hat{u}(\xi)|^2\, \mathrm{d}\xi \ = \mathrm{Im} \int_{\mathbb{R}} u^{(l+1)} \bar{u}^{(l)}\, \mathrm{d}x\,. \end{aligned}$$

- $R(t,z)$ evolves along an ODE flow: $\partial_t R(t,z) = 4iz^2 R(t,z)$, and hence we can solve the Cauchy problem (1.13) in the following way:



However, the inverse scattering transform step is rather involved and it is hard to say that this machinery can work easier than other methods. Nevertheless it offers an algorithm to solve (NLS) and we can derive much information from the formulation itself, e.g. asymptotic behaviors of the solutions.

The direct/inverse scattering transforms can be compared with the resolution of the linear Schrödinger equation via Fourier and inverse Fourier transform:

$$\begin{aligned} & i\partial_t u + u_{xx} = 0, \quad u|_{t=0} = u_0, \\ & \Rightarrow i\partial_t \hat{u}(\xi) - \xi^2 \hat{u}(\xi) = 0, \quad \hat{u}|_{t=0} = \widehat{u_0}(\xi), \\ & \Rightarrow \hat{u}(t,\xi) = e^{-i\xi^2 t}\widehat{u_0}(\xi) \Rightarrow u(t,x) = \mathcal{F}_x^{-1}(\hat{u}). \end{aligned} \tag{1.21}$$

The NLS hierarchy We will see that (NLS) ($d = 1$, $\kappa = 1$, $p = 3$) is a Hamiltonian flow, in the NLS hierarchy.
Let M_0 be the phase space, which is an infinite-dimensional real linear space with complex coordinates defined by pairs of functions $u(x), \bar{u}(x)$ in $\mathcal{S}(\mathbb{R})$. Let $F : M_0 \mapsto \mathbb{R}$ be an observable on M_0, and

$$\delta F(u,\bar{u}) = \int_{-\infty}^{\infty} \Big(\frac{\delta F}{\delta u}\delta u + \frac{\delta F}{\delta \bar{u}}\delta \bar{u} \Big)\, \mathrm{d}x\,.$$

We define a Poisson structure on the set of smooth real-analytic functionals on the phase space M_0:

$$\{F, G\} = i\int_{-\infty}^{\infty} \Big(\frac{\delta F}{\delta u}\frac{\delta G}{\delta \bar{u}} - \frac{\delta F}{\delta \bar{u}}\frac{\delta G}{\delta u} \Big)\, \mathrm{d}x\,,$$

We define correspondingly the symplectic form on the phase space M_0

$$\Omega = i\int_{-\infty}^{\infty} d\bar{u} \wedge du \ \mathrm{d}x\,.$$

Each observable H will give risse to a one-parameter group of transformations on M_0 defined by Hamilton's equations of motion

$$\begin{aligned} \frac{\partial u}{\partial t} &= \{H, u\} = -i\frac{\delta H}{\delta \bar{u}}, \\ \frac{\partial \bar{u}}{\partial t} &= \{H, \bar{u}\} = i\frac{\delta H}{\delta u}. \end{aligned}$$

This functional H is called the corresponding Hamiltonian.

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[06.05.2022]

Consider the following Hamiltonians in the NLS hierarchy

$$\begin{aligned} H_0 &= \int_{\mathbb{R}} |u|^2 \, \mathrm{d}x \,, \\ H_1 &= \frac{1}{2}\frac{1}{i} \int_{\mathbb{R}} u\partial_x \bar{u} \, \mathrm{d}x \, = \frac{1}{2}\mathrm{Im} \int_{\mathbb{R}} u\partial_x \bar{u} \, \mathrm{d}x \,, \\ H_2 &= \frac{1}{4} \int_{\mathbb{R}} \left(|\partial_x u|^2 + |u|^4\right) \mathrm{d}x \,, \\ H_3 &= \frac{1}{8}\mathrm{Im} \int_{\mathbb{R}} \left(\partial_x u \partial_{xx}\bar{u} + 3|u|^2 u\partial_x \bar{u}\right) \mathrm{d}x \,, \quad \cdots \end{aligned}$$

where H_k are the coefficients in the expansion of $\ln T^{-1}(z)$ as $|z| \to \infty$. The even ones are even with respect to complex conjugation and have a positive definite principal part, which are referred to as energies. The odd ones are odd if we replace u by $\bar{u}$, which are referred to as momenta. We remark that H_0, H_1, H_2 are the mass, momentum and energy functionals (up to some scaling) of (NLS).
These Hamiltonians generate the corresponding Hamiltonian flows as follows **(Exercises)**:

- H_0 generates the phase shifts: $u(t) = e^{-it}u_0$;
- H_1 generates the group of translations: $u(t,x) = u_0(x+2t)$;
- H_2 generates the (rescaled) defocusing cubic NLS flow (NLS);
- H_3 generates the defocusing mKdV flow mKdV.

These Hamiltonian flows are all commuting and $\{H_j, H_k\} = 0$. All H_j are conservation laws of the Hamiltonian flows, in particular of the NLS flow generated by H_2:

$$\frac{dF}{dt} = \int_{-\infty}^{\infty} \Big(\frac{\delta F}{\delta u}\frac{\partial u}{\partial t} + \frac{\delta F}{\delta \bar{u}}\frac{\partial \bar{u}}{\partial t}\Big) \, \mathrm{d}x \, = -\{F, H\}.$$

Hamiltonian structure A Hamiltonian structure can be easily derived from the existence of a Lagrangian.
Recall that in the classical setting $L = L(q, \dot{q}, t) : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}$, and the Euler-Lagrangian equation

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}}) - \frac{\partial L}{\partial q} = 0$$

can be written as
$$\dot{p} = \frac{\partial L}{\partial q}, \text{ where } p = \frac{\partial L}{\partial \dot{q}}.$$
If we further assume $L = L(q, \dot{q}, t)$ to be convex with respect to the second argument $\dot{q}$, then we can define the Hamiltonian $H = H(p, q, t)$: $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}$ by Legendre transform (with respect to the second argument)
$$H(p) = p\dot{q} - L(\dot{q}),$$
where for any given p, $\dot{q} = \dot{q}(p)$ denotes the maximum point of the map $p\dot{q} - L(\dot{q})$ such that $p = \frac{\partial L}{\partial \dot{q}}$. The Euler-Lagrangian equation is "transformed" into the Hamilton equations
$$\dot{p} = -\frac{\partial H}{\partial q},$$
$$\dot{q} = \frac{\partial H}{\partial p}.$$
Here we may simply take the Hamiltonian as
$$H_{NLS} = \frac{i}{2}(\bar{u}\partial_t u - u\partial_t \bar{u}) - L_{NLS}.$$
See [Sulem-Sulem 1999 The nonlinear Schrödinger equations, Section 2.1] for more details.

1.1.5 Solitary wave

"A solitary wave is a wave that travels at a constant velocity without changing its shape." Specially, let $e^{it}Q(x)$ be a solitary wave of the Schrödinger equation (NLS), with $Q(x)$ satisfying the elliptic equation
$$\Delta Q - Q = \kappa |Q|^{p-1} Q, \quad \kappa = -1, \quad Q \in H^1(\mathbb{R}^d). \tag{1.22}$$
We have taken $\kappa = -1$ the focusing case, otherwise in the defocusing case there exists only trivial solution: We test (1.22) with $\kappa = 1$ by $\bar{Q}$ to get
$$0 \geqslant -\int_{\mathbb{R}^d} |\nabla Q|^2 \,\mathrm{d}x = \int_{\mathbb{R}^d} (1 + |Q|^{p-1})|Q|^2 \,\mathrm{d}x \geqslant 0 \Rightarrow Q = 0 \in H^1(\mathbb{R}^d).$$
Hence solitary waves only exist in focusing case.
Then by symmetries in Subsection 1.1.3 we know that the following general solitary wave solution travels along the line $x = x_0 + 2vt$:
$$e^{i\lambda^{-2}t + ix\cdot v - i|v|^2 t + i\theta} Q_\lambda(x - x_0 - 2vt), \quad \theta \in \mathbb{R},\ x_0 \in \mathbb{R}^d,\ v \in \mathbb{R}^d, \tag{1.23}$$

with $Q_\lambda(x) = \frac{1}{\lambda^{\frac{2}{p-1}}} Q(\frac{x}{\lambda})$, $0 \neq \lambda \in \mathbb{R}$.
We remark here that for any $r \in [1, \infty]$, the L^r-norm of the initial datum Q is preserved by the solution $e^{it}Q$:

$$\|e^{it}Q\|_{L^r(\mathbb{R}^d)} = \|Q\|_{L^r(\mathbb{R}^d)},\ \forall t \in \mathbb{R}\,.$$

This phenomenon is totally different from the linear Schrödinger equation where the estimate (1.2) shows that $\|u(t,x)\|_{L^\infty_x(\mathbb{R}^d)}$ vanishes as $t \to \infty$. Hence the existence of the solitary waves describes a balance between the (linear) dispersion and the nonlinearity and they neither decay nor develop singularities.
If $d = 1$, then (1.22) is an ODE and one has an explicit solution (unique up to translation and sign change)

$$Q(x) = \Big(\frac{p+1}{2}\ \mathrm{sech}^2\big(\frac{p-1}{2}x\big)\Big)^{\frac{1}{p-1}} \in H^1(\mathbb{R}^1), \quad \mathrm{sech}\,(x) = \frac{2}{e^x + e^{-x}}. \tag{1.24}$$

If $p = 3$ and we consider the rescaled (NLS): $i\partial_t u + \partial_{xx} u = -2|u|^2 u$, then the spectrum of the Lax operator $L(u) := \begin{pmatrix} i\partial_x & -iu \\ -i\bar{u} & -i\partial_x \end{pmatrix}$ with $u = \mathrm{sech}\,(x)$ consists of the real axis (as the continuous spectrum) and two eigenvalues $z = \pm i$.
We will see that for any $d \geqslant 2$, in the energy-subcritical case (i.e. $1 < p < 1 + \frac{4}{d-2}$), there exists a unique *positive radial* H^1 solution (up to translation) of (1.22) in Proposition 4.5, Subsection 4.1. This unique solution is called the *ground state* and the corresponding solution $u(t,x) = e^{it}Q$ of (NLS) is the ground state standing wave and is often called *soliton*.
We will show the orbital stability result of the solitons in the mass-subcritical case (i.e. $1 < p < 1 + \frac{4}{d}$) in Subsection 4.3. Another interesting observation in [Weinstein 1983 CMP] (we do not give a proof here): For $1 < p < 1 + \frac{4}{d-2}$, Q can be characterized as the minimum point ψ (up to some scaling) of the functional

$$J(u) = \frac{\|\nabla u\|_{L^2}^{p_1} \|u\|_{L^2}^{p_2}}{\|u\|_{L^{p+1}}^{p+1}}, \quad u \in H^1(\mathbb{R}^d).$$

for some p_1, p_2 (depending on p, d) such that $p_1 + p_2 = p + 1$. Hence the minimisation inequality

$$\frac{d^2}{d\varepsilon^2}|_{\varepsilon=0} J(\psi + \varepsilon\eta) \geqslant 0, \quad \forall \eta \in C_0^\infty(\mathbb{R}^d)$$

will give some "positivity property" of the linearised operator around Q in (1.22), which will help to show the stability result in Subsection **??**.

There are also other solutions (not necessarily positive or radial) than the ground state for (1.22) when $d \geqslant 2$ which are called bound states and we will not discuss them in this lecture.

1.2 Other related models

1.2.1 Hydrodynamic NLS

Let u have no zeros. Then by virtue of the so-called Madelung transform

$$u(t, \sqrt{2}x) = \sqrt{\rho(t,x)}e^{i\phi(t,x)}$$

(this is possible if u is away from zero), we obtain the following system for the unknown (ρ, v) with $v = \nabla\phi$ from (NLS):

$$\begin{cases} \partial_t \rho + \operatorname{div}_x(\rho v) = 0, \\ \partial_t v + v \cdot \nabla_x v + \nabla_x(\kappa \rho^{\frac{p-1}{2}}) = \nabla_x\big(\frac{\Delta\sqrt{\rho}}{2\sqrt{\rho}}\big). \end{cases} \tag{1.25}$$

(**Exercise**.) The above system is referred to as the hydrodynamic form of (NLS) because of its similarity of the compressible Euler system. The ρ-equation can be viewed as the continuity equation and the v-equation can be viewed as the momentum equation with an additional quantum pressure $\nabla_x\big(\frac{\Delta\sqrt{\rho}}{2\sqrt{\rho}}\big)$. The system (1.25) is also called quantum Euler equation or Euler-Korteweg equation in the theories of quantum fluids and Korteweg fluids.

1.2.2 Maxwell equation

Vacuum case The propagation of an electromagnetic wave (e.g. a laser pulse) in vacuum is governed by Maxwell's equations:

$$\operatorname{curl}\mathcal{E} = -\partial_t\mathcal{B}, \quad \operatorname{curl}\mathcal{H} = \partial_t\mathcal{D}, \quad \operatorname{div}\mathcal{D} = 0, \quad \operatorname{div}\mathcal{B} = 0, \tag{1.26}$$

where $t \in \mathbb{R}$, $x \in \mathbb{R}^3$ are the time and space variables, $\mathcal{E}, \mathcal{H} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ are the electric and the magnetic fields respectively, and $\mathcal{D}, \mathcal{B} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ are the electric and magnetic induction fields respectively. In vacuum, $\mathcal{D}, \mathcal{B}$ are related to $\mathcal{E}, \mathcal{H}$ by the constitutive relations

$$\mathcal{D} = \epsilon_0\mathcal{E}, \quad \mathcal{B} = \mu_0\mathcal{H},$$

where ϵ_0, μ_0 are vacuum permittivity and permeability respectively.
Then we derive the wave equation for the electric field:

$$\partial_t^2\mathcal{E}_j - c^2\Delta\mathcal{E}_j = 0, \quad j = 1, 2, 3,$$

where $c = (\epsilon_0\mu_0)^{-\frac{1}{2}}$ is the speed of light in vacuum. (**Exercise**.)
We look for the time-harmonic solution of the form

$\mathcal{E}_j(t,x) = e^{-i\omega_0 t}E(x)+$ c.c., with "c.c." standing for the complex conjugate,

where $E(x)$ satisfies the scalar linear Helmholtz equation

$$\Delta E + k_0^2 E = 0, \text{ with } k_0^2 = \frac{\omega_0^2}{c^2}.$$

We look for solutions of the Helmholtz equation of the form

$$E(x) = e^{ik_0x_3}\psi(x),$$

where ψ is the electric-field envelope (or amplitude) and satisfies

$$\partial_{x_3x_3}\psi + 2ik_0\partial_{x_3}\psi + (\partial_{x_1x_1} + \partial_{x_2x_2})\psi = 0.$$

In the paraxial approximation (i.e. ψ is slowly-varying in x_3-direction, compared with the carrier oscillation $e^{ik_0x_3}$), we can neglect $\partial_{x_3x_3}\psi$[3] (mathematically questionable) and the above equation for ψ becomes the linear Schrödinger equation

$$2ik_0\partial_{x_3}\psi + (\partial_{x_1x_1} + \partial_{x_2x_2})\psi = 0.$$

Dielectric medium
When an electric field applies on a dielectric medium, it induces an additional electric field, which is called the polarization field. Hence the electrical induction field $\mathcal{D}$ in a dielectric medium becomes the sum of the original electric field and the polarization field:

$$\mathcal{D} = \epsilon_0\mathcal{E} + \mathcal{P}.$$

For simplicity let us assume that $\mathcal{E}, \mathcal{D}, \mathcal{P} \in \mathbb{R}$ (that is, the electric field is linearly polarized: $(\mathcal{E}, 0, 0)$). At low intensities, the dependence of $\mathcal{P}$ on $\mathcal{E}$ is linear:

$$\mathcal{P} = \mathcal{P}_{\text{lin}} = \epsilon_0\chi^{(1)}(\omega_0)\mathcal{E},$$

[3] Consider the plane waves $E = E_c e^{ik_1x_1+ik_2x_2+ik_3x_3}$ with $k_1^2+k_2^2+k_3^2 = k_0^2$ for the scalar linear Helmholtz equation, such that $\psi = E_c e^{ik_1x_1+ik_2x_2+i(k_3-k_0)x_3}$. Then for paraxial plane waves $k_3 - k_0 \ll k_0$,

$$\frac{|\psi_{33}|}{|k_0\psi_3|} = \frac{(k_0-k_3)^2}{k_0|k_0-k_3|} = \frac{|k_0-k_3|}{k_0} \ll 1;$$
$$\frac{|\psi_{33}|}{|\psi_{11}+\psi_{22}|} = \frac{(k_0-k_3)^2}{|k_0^2-k_3^2|} = \frac{|k_0-k_3|}{(k_0+k_3)} \ll 1.$$

where $\chi^{(1)}$ is the first-order optical susceptibility. Hence the scalar Helmholtz equation in a linear dielectric becomes

$$\Delta E + k_0^2 E = 0, \quad \text{with } k_0^2 = \frac{\omega_0^2}{c^2} n_0^2, \tag{1.27}$$

where $n_0 = \sqrt{1 + \chi^{(1)}}$ is the (linear) index of refraction of the medium.
As $\mathcal{E}$ increases, the dependence becomes nonlinear and in the weakly nonlinear regime we have

$$\mathcal{P} = \mathcal{P}_{\text{lin}} + \mathcal{P}_{\text{nl}}, \quad \mathcal{P}_{\text{nl}} \approx \chi^{(3)}(\omega_0)\mathcal{E}^3 \ll \mathcal{P}_{\text{lin}},$$

where $\chi^{(2j)} = 0$ in isotropic materials. Let $\mathcal{E} = e^{-i\omega_0 t} E + c.c.$, then (we neglect the part with frequency $3\omega_0$)

$$\mathcal{P}_{\text{nl}} \approx \chi^{(3)}\big(3|E|^2 E e^{-i\omega_0 t} + c.c.\big) = 3\chi^{(3)}|E|^2\mathcal{E} =: 4\epsilon_0 n_0 n_2 |E|^2 \mathcal{E},$$

where we defined the Kerr coefficient

$$n_2 = \frac{3\chi^{(3)}}{4\epsilon_0 n_0}.$$

Therefore,

$$\mathcal{D} = \epsilon_0 n^2 \mathcal{E}, \text{ with } n^2 = n_0^2\big(1 + \frac{4n_2}{n_0}|E|^2\big).$$

Hence if we formally replace n_0^2 by n^2 in the scalar nonlinear Helmholtz equation (1.27), we get the scalar nonlinear Helmholtz equation for the propagation of a linearly-polarized laser beam in a Kerr medium ($\mathcal{E}$ does not necessarily satisfy the wave equation $n^2\partial_t^2\mathcal{E} - c^2\Delta\mathcal{E} = 0$)

$$\Delta E + k^2 E = 0, \quad \text{with } k^2 = k_0^2\big(1 + \frac{4n_2}{n_0}|E|^2\big).$$

We substitute $E = e^{ik_0 x_3}\psi$ into the above equation and apply the paraxial approximation ($\psi_{x_3 x_3} \ll k_0 \psi_{x_3}$), to derive the nonlinear Schrödinger equation (NLS) for ψ:

$$2ik_0\partial_{x_3}\psi + (\partial_{x_1x_1} + \partial_{x_2x_2})\psi + k_0^2\frac{4n_2}{n_0}|\psi|^2\psi = 0. \tag{1.28}$$

(**Exercise**: Use the symmetry property to rewrite the above equation into the standard form (NLS).)
To conclude, the above (NLS) is the leading order model for paraxial propagation of intense linearly-polarized continuous wave laser beams in a homogeneous Kerr medium, in which ψ is the slowly-varying amplitude of the electric field, x_3 is the direction of propagation.

1.2.3 Water waves

The details of this subsection can be found in [Zakharov 1968 Stability of periodic waves of finite amplitude on the surface of a deep fluid], and we just sketch the ideas here.
We consider the potential flow of an ideal fluid in the domain $\{(x,y,z) \mid (x,y) \in \mathbb{R}^2,\ z \in (-\infty, \eta]\}$, in the presence of a gravity field. Let $\eta = \eta(x,y,t)$ be the shape of the surface of the fluid and let $\phi = \phi(x,y,z,t)$ be the hydrodynamic potential (that is, the velocity field $v = (\partial_x, \partial_y, \partial_z)^T \phi$). Denote simply $\nabla = \nabla_{x,y}$, $\Delta = \Delta_{x,y}$ on the xy-plane. Then the dynamic of the ideal fluid is governed by the Laplace's equation for ϕ:

$$\Delta\phi + \partial_{zz}\phi = 0, \quad (x,y) \in \mathbb{R}^2, \quad z \in (-\infty, \eta),$$

together with the boundary conditions at the surface (neglecting the surface tension):

$$\begin{aligned} &\text{kinetic b.c.: } \partial_t \eta = \partial_z|_{z=\eta}\phi - \nabla\eta \cdot \nabla\phi|_{z=\eta}, \\ &\text{dynamic b.c.: } \partial_t\phi|_{z=\eta} + g\eta = -\frac{1}{2}\big(|\nabla\phi|^2 + (\partial_z\phi)^2\big)|_{z=\eta}, \end{aligned} \tag{1.29}$$

where g is the gravity acceleration and the condition at infinity:

$$\phi \to 0, \text{ as } z \to -\infty.$$

We introduce $\psi = \psi(x,y,t) = \phi\big(x,y,z = \eta(x,y),t\big)$, then $\partial_t\psi = \partial_t\phi + \partial_t\eta\partial_z|_{z=\eta}\phi$. Hence the above boundary conditions (1.29) become the evolutionary equations for (η, ψ):

$$\begin{cases} \partial_t\eta = \partial_z|_{z=\eta}\phi - \nabla\eta \cdot \nabla\phi|_{z=\eta}, \\ \partial_t\psi + g\eta = -\frac{1}{2}|\nabla\phi|^2 + \frac{1}{2}(\partial_z\phi)^2|_{z=\eta} - \partial_z\phi(\nabla\eta \cdot \nabla\phi)|_{z=\eta}, \end{cases} \tag{1.30}$$

where ϕ solves the Laplace's equation in $\mathbb{R}^2 \times (-\infty, \eta]$ together with the boundary conditions $\phi|_{z=\eta} = \psi$, $\phi \to 0$ as $|z| \to \infty$. We remark that the system (1.30) are Hamiltonian's equations (details can be found in [Zakharov 1968]):

$$\partial_t\eta = \frac{\delta H}{\delta\psi}, \quad \partial_t\psi = -\frac{\delta H}{\delta\eta}, \tag{1.31}$$

where H is the Hamiltonian

$$H = \frac{1}{2}\int_{\mathbb{R}^2}\int_{-\infty}^{\eta}\Big(|\nabla\phi|^2 + (\partial_z\phi)^2\Big)dz\,\mathrm{d}x\ \mathrm{d}y + \frac{1}{2}g\int_{\mathbb{R}^2}\eta^2\,\mathrm{d}x\ \mathrm{d}y\,,$$

and ψ is the generalized coordinate and η is the generalized momentum.

We take the Fourier transform of η, ψ on the xy-plane, and linearize (1.30) under the small amplitude assumption :

$$\begin{cases} \partial_t \hat{\eta} - |k|\hat{\psi} = 0, \\ \partial_t \hat{\psi} + g\hat{\eta} = 0, \end{cases} \tag{1.32}$$

where we used the expansion for ϕ: $\hat{\phi}(k) = e^{|k|z}\hat{\psi}(k) + O(|\eta|)$. Therefore we derive the dispersion relation

$$\omega = \omega(k) = \sqrt{g|k|}, \quad k = (k_x, k_y).$$

We introduce $a = a(k) = \frac{1}{\sqrt{2}}\big(\hat{\eta}(k)(\frac{\omega(k)}{|k|})^{\frac{1}{2}} + i\hat{\psi}(k)(\frac{|k|}{\omega(k)})^{\frac{1}{2}}\big)$ [4] to diagnize the linear equations (1.32):

$$\partial_t a(k) + i\omega(k)a(k) = 0.$$

Then Hamilton's equation (1.31) becomes a single equation $\partial_t a(k) = -i\frac{\delta H}{\delta \bar{a}(k)}$, where $H = \int_{\mathbb{R}^2} \omega(k)a(k)\bar{a}(k)dk + O(|a|^3)$, and we write (not obviously)

$$\begin{aligned} \partial_t a(k) + i\omega(k)a(k) = &-i\int \Big(V(-k, k_1, k_2)a(k_1)a(k_2)\delta(k - k_1 - k_2) \\ &+ 2V(-k_1, k, k_2)\bar{a}(k_2)a(k_1)\delta(k - k_1 + k_2) + V(k, k_1, k_2)\bar{a}(k_1)\bar{a}(k_2)\delta(k + k_1 + k_2)\Big)dk_1dk_2 \\ &- i\int W(k, k_1, k_2, k_3)\bar{a}(k_1)a(k_2)a(k_3)\delta(k + k_1 - k_2 - k_3)dk_1dk_2dk_3 + O(|a|^4), \end{aligned}$$

where $V(k, k_1, k_2)$, $W(k, k_1, k_2, k_3)$ depend on their arguments explicitly (details omitted here) and δ is the delta function.
Under the small amplitude assumption we write $a(k)$ as

$$a(k) = \big(A(k,t) + f(k,t)\big)e^{-i\omega(k)t},$$

where $A(k,t)$ changes slowly in comparison with f while $|f| \ll |A| \ll 1$. Assuming A constant when f varies, we integrate the equation of a with respect to the time to arrive at the following expression for f (up to $|A|^3$):

$$\begin{aligned} f = &-\int \Big(V(-k, k_1, k_2)\frac{\exp it\big(\omega(k) - \omega(k_1) - \omega(k_2)\big)}{\omega(k) - \omega(k_1) - \omega(k_2)}A(k_1)A(k_2)\delta(k - k_1 - k_2) \\ &+ 2V(-k_1, k, k_2)\frac{\exp it\big(\omega(k) + \omega(k_1) - \omega(k_2)\big)}{\omega(k) + \omega(k_1) - \omega(k_2)}\bar{A}(k_2)A(k_1)\delta(k - k_1 + k_2) \\ &+ V(k, k_1, k_2)\frac{\exp it\big(\omega(k) + \omega(k_1) + \omega(k_2)\big)}{\omega(k) + \omega(k_1) + \omega(k_2)}\bar{A}(k_1)\bar{A}(k_2)\delta(k + k_1 + k_2)\Big)dk_1dk_2. \end{aligned}$$

[4] Then $\bar{a}(-k) = \frac{1}{\sqrt{2}}\big(\hat{\eta}(k)(\frac{\omega(k)}{|k|})^{\frac{1}{2}} - i\hat{\psi}(k)(\frac{|k|}{\omega(k)})^{\frac{1}{2}}\big)$.

In the evolutionary equation for A we retain only the terms proportional to Af which contain the most slowly varying exponents, such that

$$\partial_t A = -i \int T(k, k_1, k_2, k_3)\delta(k + k_1 - k_2 - k_3) \\ \times \exp it\big(\omega(k) + \omega(k_1) - \omega(k_2) - \omega(k_3)\big)\bar{A}(k_1)A(k_2)A(k_3)dk_1dk_2dk_3,$$

where T depends explicitly on its arguments (details omitted here).
Under the narrow wave packet assumption: $|\xi| = |k - k_0| \ll |k_0|$, we expand $\omega(k)$ in powers of $\xi = (\xi_1, \xi_2)$ (ξ_1, ξ_2 are the projections of the vector ξ along and perpendicular to the vector $k - k_0$ respectively) around k_0 as follows:

$$\omega(k) = \omega(k_0) + \xi_1 c_g + \frac{1}{2}(\lambda_1\xi_1^2 + \lambda_2\xi_2^2) + \cdots ,$$
$$c_g = \partial_k|_{k=k_0}\omega, \quad \lambda_1 = \partial_k^2|_{k=k_0}\omega, \quad \lambda_2 = \frac{c_g}{k_0}.$$

Finally we introduce

$$b(k, t) = A(k, t)e^{i(\xi_1 c_g + \frac{1}{2}(\chi_1\xi_1^2 + \chi_2\xi_2^2))t},$$

and verify that its inverse Fourier transform $\check{b}(x_1, x_2, t)$ with respect to $\xi = (\xi_1, \xi_2)$ satisfies the cubic nonlinear Schrödinger equation (NLS):

$$\partial_t\check{b} + c_g\partial_1\check{b} - \frac{i}{2}\big(\lambda_1\partial_{11}\check{b} + \lambda_2\partial_{22}\check{b}\big) = -i\kappa|\check{b}|^2\check{b}. \tag{1.33}$$

[06.05.2022]
[09.05.2022]

2 Wellposedness

Definition 2.1 (LWP & GWP). *The Cauchy problem* (NLS)

$$\begin{cases} i\partial_t u + \Delta u = \kappa |u|^{p-1}u, \\ u|_{t=0} = u_0(x), \end{cases}$$

is said to be locally well-posed LWP in $H^s(\mathbb{R}^d)$ if for any initial data $u_0 \in H^s(\mathbb{R}^d)$, there exists a positive time $T > 0$ and a unique solution $u \in C([-T,T]; H^s(\mathbb{R}^d))$ of (NLS) *such that there exists a neighbourhood U of u_0 in $H^s(\mathbb{R}^d)$ and the flow map*

$$\Phi : U \mapsto H^s(\mathbb{R}^d), \quad u_0 \mapsto u(t, \cdot)$$

is continuous for any $t \in (-T, T)$.
We say that (NLS) *is globally well-posedness GWP in $H^s(\mathbb{R}^d)$ if the above holds on any time interval $[-T,T]$, $T > 0$.*

Recall the famous Hadamard's example of the ill-posed Cauchy problem for the Laplace equation:

$$\begin{cases} v_{tt} + v_{xx} = 0, \\ v|_{t=0} = 0, \quad v_t|_{t=0} = f(x). \end{cases}$$

Exercise. Show the existence and the uniqueness results of the solution for the above Cauchy problem with the following initial data sequence

$$(v, v_t)|_{t=0} = (0, f_n) = (0, e^{-\sqrt{n}} n \sin(nx)) \to 0 \text{ as } n \to \infty, \text{ in any } C_b^k(\mathbb{R}),\ k \in \mathbb{N},$$

while the continuity of the flow map fails.
We will show the well-posedness results in the subcritical cases for (NLS) in this section. Recalling the Duhamel formula (1.5), we would like to apply the fixed point theorem to show the unique existence of the solution $u \in X_T \subset C([-T,T]; H^s(\mathbb{R}^d))$. The choice of the functional space X_T is crucial and we have to make sure that the linear map $u_0 \mapsto S(t)u_0$ is from $H^s(\mathbb{R}^d)$ to X_T while the (nonlinear) map $u \mapsto \int_0^t S(t-t')(f(u)(t'))\,\mathrm{d}t'$, $f(u) = \kappa|u|^{p-1}u$ is from X_T to X_T. Finally we can choose the time T small enough such that these maps are contraction mappings and hence the fixed point theorem works. The mass/energy conservation laws then imply GWP in the L^2/H^1 framework respectively.

2.1 Strichartz estimates

2.1.1 Preliminary estimates

By Stone's theorem, for the selfadjoint operator Δ on the Hilbert space $H = L^2(\mathbb{R}^d)$ with the domain $D(\Delta) = H^2(\mathbb{R}^d)$, there exists a unique strongly continuous unitary group $S(t) = e^{it\Delta}$ on H such that

$$\frac{d}{dt}\Big|_{t=0} S(t)\phi = i\Delta\phi, \quad \forall \phi \in D(\Delta) = H^2(\mathbb{R}^d).$$

Recalling the solution (1.1) of the linear NLS equation, we have

$$S(t)g = e^{it\Delta}g = K_t * g = (4\pi it)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{i\frac{|x-y|^2}{4t}} g(y)dy. \tag{St}$$

We rewrite the Duhamel formula (1.5) as

$$u(t,x) = S(t)u_0 - i\kappa \int_0^t S(t-s)\big(|u(s)|^{p-1}u(s)\big)ds. \tag{Duhamel}$$

Recall (1.2) (it is obvious from (St))

$$\|S(t)u_0\|_{L^\infty(\mathbb{R}^d)} \leqslant |4\pi t|^{-\frac{d}{2}} \|u_0\|_{L^1(\mathbb{R}^d)}, \tag{2.34}$$

which implies that if $u_0 \in L^1(\mathbb{R}^d)$, then the solution $S(t)u_0$ of the linear Schrödinger equation decays of rate $|t|^{-\frac{d}{2}}$ as $|t| \to \infty$. This is exactly the dispersion phenomenon. Since $S(t)$ is unitary, then

$$\|S(t)u_0\|_{L^2(\mathbb{R}^d)} = \|u_0\|_{L^2(\mathbb{R}^d)}. \tag{2.35}$$

Recall the Riesz-Thorine interpolation theorem:

Theorem 2.1. *Let $1 \leqslant p_0 \neq p_1 \leqslant \infty$, $1 \leqslant q_0 \neq q_1 \leqslant \infty$. If $T \in \mathcal{L}(L^{p_j}(\mathbb{R}^d), L^{q_j}(\mathbb{R}^d))$ be the linear operator from $L^{p_j}(\mathbb{R}^d)$ to $L^{q_j}(\mathbb{R}^d)$ with the operator norm M_j, $j = 0, 1$, then for any $0 \leqslant \theta \leqslant 1$,*

$$T \in \mathcal{L}(L^{p_\theta}(\mathbb{R}^d), L^{q_\theta}(\mathbb{R}^d)), \quad \frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \ \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1},$$

with the operator norm $M_\theta \leqslant M_0^{1-\theta} M_1^\theta$.

Hence we derive from (2.34)-(2.35) that

Proposition 2.1. *Let $S(t)$ be the unitary map defined by* (St). *Then $S(t)$ is a linear map from $L^{r'}(\mathbb{R}^d)$ to $L^r(\mathbb{R}^d)$ for any $r \in [2, \infty]$ (with $\frac{1}{r} + \frac{1}{r'} = 1$) such that*

$$\|S(t)u_0\|_{L^r(\mathbb{R}^d)} \leqslant |4\pi t|^{-\frac{d}{2}+\frac{d}{r}} \|u_0\|_{L^{r'}(\mathbb{R}^d)}, \quad \forall t \neq 0. \tag{2.36}$$

Recall Proposition 2.1 that the Schrödinger group $S(t)$ maps $L^{r'}(\mathbb{R}^d)$ to $L^r(\mathbb{R}^d)$, $r \geqslant 2$ with the operator norm $|4\pi t|^{-(\frac{d}{2}-\frac{d}{r})}$. It is also obvious from the definition of $S(t)$: $\widehat{S(t)g}(\xi) = e^{-it|\xi|^2}\widehat{g}(\xi)$ and the definition of H^s-norm (1.6) that

$$\|S(t)g\|_{H^s(\mathbb{R}^d)} = \|g\|_{H^s(\mathbb{R}^d)}, \quad \forall s \in \mathbb{R}\,.$$

Remark 2.1. ***(Exercise.)*** *The operator $S(t)$, $t > 0$ does not map*

- *from $L^2(\mathbb{R}^d)$ to $L^r(\mathbb{R}^d)$ or from $L^r(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$ for $r \neq 2$;*
- *from $L^r(\mathbb{R}^d)$ to $L^r(\mathbb{R}^d)$ for $r \neq 2$;*
- *from $L^r(\mathbb{R}^d)$ to $L^{r_1}(\mathbb{R}^d)$ for any $r > 2$;*
- *from $H^s(\mathbb{R}^d)$ to $H^{s'}(\mathbb{R}^d)$ for $s' > s$.*

The heat semigroup $e^{t\Delta} = A_t *$ with $A_t = (4\pi t)^{-\frac{d}{2}}e^{-|\cdot|^2/4t} \in L^1(\mathbb{R}^d)$, $t > 0$ maps from L^r to L^r and from H^s to $H^{s'}$, for any $r \in [1, \infty]$ and $s' \geqslant s$.

2.1.2 Strichartz estimates

We will show the well-known Strichartz estimates for $S(t)$ in this subsection.

Theorem 2.2. *[Strichartz estimates for the Schrödinger group] Let (q, r) be admissible exponent pair, i.e.*

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2}, \quad 2 \leqslant q, r \leqslant \infty, \quad (q, r, d) \neq (2, \infty, 2). \tag{2.37}$$

Then for any admissible exponent pairs $(q, r), (\widetilde{q}, \widetilde{r})$, we have the following homogeneous Strichartz estimate (for the solution $S(t)u_0$ of the homogeneous problem $i\partial_t u + \Delta u = 0, u|_{t=0} = u_0$)

$$\|S(t)u_0\|_{L^q_t L^r_x(\mathbb{R}\times\mathbb{R}^d)} \leqslant C(q, r, d)\|u_0\|_{L^2_x(\mathbb{R}^d)}, \tag{2.38}$$

and the inhomogeneous Strichartz estimate (for the solution $\int_0^t S(t-t')f(t')\,\mathrm{d}t'$ of the inhomogeneous problem $i\partial_t u + \Delta u = f, u|_{t=0} = 0$)

$$\left\|\int_0^t S(t-t')f(t')\,\mathrm{d}t'\right\|_{L^q_t L^r_x(\mathbb{R}\times\mathbb{R}^d)} \leqslant C(q, r, \widetilde{q}, \widetilde{r}, d)\|f\|_{L^{\widetilde{q}'}_t L^{\widetilde{r}'}_x(\mathbb{R}\times\mathbb{R}^d)}, \tag{2.39}$$

where $\frac{1}{q} + \frac{1}{q'} = 1$ and the space-time norm $\|\cdot\|_{L^q_t L^r_x(\mathbb{R}\times\mathbb{R}^d)}$ is defined to be

$$\|g\|_{L^q_t L^r_x(\mathbb{R}\times\mathbb{R}^d)} = \Big\|\|g(t,\cdot)\|_{L^r_x(\mathbb{R}^d)}\Big\|_{L^q_t(\mathbb{R})}.$$

In the above the time definition domain $\mathbb{R}$ can be replaced by any time interval $[-T, T]$, $T > 0$.

Remark 2.2. • *By virtue of* (2.38)*, if* $u_0 \in L^2$*, then* $S(t)u_0 \in L^r$ *with*

$$2 \leqslant r \leqslant \infty \text{ if } d = 1, \quad 2 \leqslant r < \infty \text{ if } d = 2, \quad 2 \leqslant r \leqslant \frac{2d}{d-2} \text{ if } d \geqslant 3,$$

for almost all $t \in \mathbb{R}$ *and the norm* $\|S(t)u_0\|_{L^r}$*,* $r > 2$ *decays faster than* $|t|^{-\frac{1}{q}}$ *for almost all the time. This indicates the smooth and decay effects of* $S(t)$*. On the other side we know that there exists* $u_0 \in L^2$ *and* $t \in \mathbb{R}$ *such that* $S(t)u_0 \notin L^r$ *since the operator* $S(t)$ *is not a map from* L^2 *to* L^r*.*

- *There are infinite many admissible exponent pairs and we can always have the trivial case* $(q, r) = (\infty, 2)$ *and the particular case* $q = r = 2(d+2)/d$*. In the case* $(q, r) = (\infty, 2)$*, we have indeed also the continuity in* $L^2(\mathbb{R}^d)$*: If* $u_0 \in L^2(\mathbb{R}^d)$*, then* $S(t)u_0 \in C(\mathbb{R}; L^2_x(\mathbb{R}^d))$*, and if* $f \in L^{q'}_t L^{r'}_x(\mathbb{R} \times \mathbb{R}^d)$*, then* $\int_0^t S(t-t')f(t')dt' \in C(\mathbb{R}; L^2_x(\mathbb{R}^d))$*.*

  If $d = 1$*, then* $q \in [4, \infty]$ *and*

$$\|S(t)u_0\|_{L^4_t L^\infty_x(\mathbb{R} \times \mathbb{R})} + \|S(t)u_0\|_{L^6_{t,x}(\mathbb{R} \times \mathbb{R})} \leqslant C\|u_0\|_{L^2(\mathbb{R})}.$$

- *The equality for the admissible exponent paris can be seen as follows: Let* u *be a solution of the homogeneous linear Schrödinger equation, then the rescaled solution* $u_\lambda(t, x) = u(\frac{t}{\lambda^2}, \frac{x}{\lambda})$ *with rescaled initial data* $u_{0,\lambda} = u(\frac{x}{\lambda})$*,* $\lambda > 0$ *such that*

$$\|u_\lambda\|_{L^q_t L^r_x(\mathbb{R} \times \mathbb{R}^d)} = \lambda^{(\frac{2}{q}+\frac{d}{r})}\|u\|_{L^q_t L^r_x(\mathbb{R} \times \mathbb{R}^d)}, \quad \|u_{0,\lambda}\|_{L^2_x(\mathbb{R}^d)} = \lambda^{\frac{d}{2}}\|u_0\|_{L^2_x(\mathbb{R}^d)},$$

  also solves the linear Schrödinger equation. If (2.38) *holds for* u*, then it also holds for* u_λ *for any* $\lambda > 0$ *and the only possibility is* $\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$*.*

Proof. **Step 1 From** $L^{q'}_t L^{r'}_x$ **to** $L^q_t L^r_x$ **for** $2 < q < \infty$
By use of the estimate (2.36), we know for any $-\infty \leqslant t_1 < t_2 \leqslant \infty$

$$\begin{aligned}\Big\|\int_{t_1}^{t_2} S(t-t')f(t')\,\mathrm{d}t'\Big\|_{L^q_t L^r_x(\mathbb{R} \times \mathbb{R}^d)} &\leqslant \Big\|\int_{t_1}^{t_2} (4\pi|t-t'|)^{-(\frac{d}{2}-\frac{d}{r})}\|f(t')\|_{L^{r'}_x(\mathbb{R}^d)}\,\mathrm{d}t'\Big\|_{L^q_t(\mathbb{R})} \\ &\leqslant \Big\|\int_{\mathbb{R}} (4\pi|t-t'|)^{-\frac{2}{q}}\|f(t')\|_{L^{r'}_x(\mathbb{R}^d)}\,\mathrm{d}t'\Big\|_{L^q_t(\mathbb{R})}.\end{aligned}$$

By the Hardy-Littlewood-Sobolev inequality (**Exercise**)

$$\begin{aligned}&\|g * |\cdot|^{-\alpha}\|_{L^q(\mathbb{R}^n)} \leqslant C(p, q, \alpha, n)\|g\|_{L^m(\mathbb{R}^n)}, \\ &1 + \frac{1}{q} = \frac{1}{m} + \frac{\alpha}{n}, \quad 0 < \alpha < n, \quad 1 < m < q < \infty,\end{aligned}$$

with

$$n = 1, \quad \alpha = \frac{d}{2} - \frac{d}{r} = \frac{2}{q}, \quad m = q', \ (2 < q < \infty),$$

we derive from the above inequality that for any $-\infty \leqslant t_1 < t_2 \leqslant \infty$ ($t_1 < t_2$ may be any two functions of t)

$$\Big\| \int_{t_1}^{t_2} S(t-t') f(t') \, \mathrm{d}t' \Big\|_{L_t^q L_x^r(\mathbb{R} \times \mathbb{R}^d)} \leqslant C(q,r,d) \|f\|_{L_t^{q'} L_x^{r'}(\mathbb{R} \times \mathbb{R}^d)}.$$

Step 2 From $L_t^{q'} L_x^{r'}$ to $L_t^\infty L_x^2$ for $2 < q < \infty$
We calculate for any $t \in \mathbb{R}$, $-\infty \leqslant t_1 < t_2 \leqslant \infty$

$$\begin{aligned}
&\Big\| \int_{t_1}^{t_2} S(t-t') f(t') \, \mathrm{d}t' \Big\|_{L_x^2(\mathbb{R}^d)}^2 = \Big\langle \int_{t_1}^{t_2} S(t-t') f(t') \, \mathrm{d}t' \, , \int_{t_1}^{t_2} S(t-t'') f(t'') \, dt'' \Big\rangle_{L_x^2(\mathbb{R}^d)} \\
&= \int_{t_1}^{t_2} \int_{t_1}^{t_2} \langle S(t-t') f(t'), S(t-t'') f(t'') \rangle_{L_x^2(\mathbb{R}^d)} \, dt' \, dt'' \\
&= \int_{t_1}^{t_2} \int_{t_1}^{t_2} \langle f(t'), S(t'-t'') f(t'') \rangle_{L_x^2(\mathbb{R}^d)} \, dt' \, dt'' \\
&= \int_{t_1}^{t_2} \Big\langle f(t'), \int_{t_1}^{t_2} S(t'-t'') f(t'') \, dt'' \Big\rangle_{L_x^2(\mathbb{R}^d)} dt' \\
&\leqslant \|f(t')\|_{L_{t'}^{q'} L_x^{r'}(\mathbb{R}^d)} \Big\| \int_{t_1}^{t_2} S(t'-t'') f(t'') \, dt'' \Big\|_{L_{t'}^q L_x^r(\mathbb{R} \times \mathbb{R}^d)} \\
&\leqslant C \|f\|_{L_t^{q'} L_x^{r'}(\mathbb{R} \times \mathbb{R}^d)}^2 \text{ by Step 1.}
\end{aligned}$$

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Step 3 Proof of (2.38) and from $L_t^1 L_x^2$ to $L_t^q L_x^r$ by duality
By duality, we derive (2.38) by Step 2

$$\begin{aligned}
\|S(t)u_0\|_{L_t^q L_x^r(\mathbb{R} \times \mathbb{R}^d)} &= \sup_{\|g\|_{L_t^{q'} L_x^{r'}(\mathbb{R} \times \mathbb{R}^d)} \leqslant 1} \Big| \int_{\mathbb{R}} \langle S(t)u_0, g(t) \rangle_{L_x^2(\mathbb{R}^d)} dt \Big| \\
&= \sup_{\|g\|_{L_t^{q'} L_x^{r'}(\mathbb{R} \times \mathbb{R}^d)} \leqslant 1} \Big| \int_{\mathbb{R}} \langle u_0, S(-t) g(t) \rangle_{L_x^2(\mathbb{R}^d)} dt \Big| \\
&\leqslant \sup_{\|g\|_{L_t^{q'} L_x^{r'}(\mathbb{R} \times \mathbb{R}^d)} \leqslant 1} \|u_0\|_{L_x^2(\mathbb{R}^d)} \Big\| \int_{\mathbb{R}} S(0-t') g(t') dt' \Big\|_{L^2} \\
&\leqslant C \|u_0\|_{L_x^2(\mathbb{R}^d)} \text{ by Step 2.}
\end{aligned}$$

Similarly, we can show for any $-\infty \leqslant t_1 < t_2 \leqslant \infty$,

$$\Big\| \int_{t_1}^{t_2} S(t-t') f(t') \, \mathrm{d}t' \Big\|_{L_t^q L_x^r(\mathbb{R} \times \mathbb{R}^d)}$$

$$= \sup_{\|g\|_{L_t^{q'}L_x^{r'}(\mathbb{R}\times\mathbb{R}^d)}\leqslant 1} \Big| \int_{\mathbb{R}} \Big\langle \int_{t_1}^{t_2} S(t-t')f(t')\,\mathrm{d}t', g(t) \Big\rangle_{L_x^2(\mathbb{R}^d)} dt \Big|$$
$$= \sup_{\|g\|_{L_t^{q'}L_x^{r'}(\mathbb{R}\times\mathbb{R}^d)}\leqslant 1} \Big| \int_{t_1}^{t_2} \Big\langle f(t'), \int_{\mathbb{R}} S(t'-t)g(t)dt \Big\rangle_{L_x^2(\mathbb{R}^d)} dt' \Big|$$
$$\leqslant C\|f\|_{L_t^1 L_x^2(\mathbb{R}\times\mathbb{R}^d)} \text{ by Step 2.}$$

If $(t_1, t_2) = (0, t)$, then we just take the integral intervals $(0, \infty)$ and (t', ∞) for the variables t' and t respectivly.

Step 4 Proof of (2.39) **by interpolation**

We have shown in Step 1 and Step 2 that the linear operator

$$f \mapsto \int_0^t S(t-t')f(t')dt'$$

is bounded from $L_t^{\tilde{q}'}L_x^{\tilde{r}'}$ to $L_t^{\tilde{q}}L_x^{\tilde{r}}$ and from $L_t^{\tilde{q}'}L_x^{\tilde{r}'}$ to $L_t^\infty L_x^2$. By the log-convexity of L^p-norms,

$$\|g\|_{L^{p_\theta}} \leqslant \|g\|_{L^{p_0}}^{1-\theta}\|g\|_{L^{p_1}}^{\theta}, \text{ with } \frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1},$$

the above operator is bounded from $L_t^{\tilde{q}'}L_x^{\tilde{r}'}$ to $L_t^q L_x^r$ if $2 < \tilde{q} \leqslant q \leqslant \infty$.

Similarly we have shown in Step 1 and Step 3 that the above linear operator is bounded from $L_t^{q'}L_x^{r'}$ to $L_t^q L_x^r$ and from $L_t^1 L_x^2$ to $L_t^q L_x^r$ and hence from $L_t^{\tilde{q}'}L_x^{\tilde{r}'}$ to $L_t^q L_x^r$ if $1 \leqslant \tilde{q}' \leqslant q' < 2$.

These two cases complete the estimate (2.39) for $2 < q \leqslant \infty$.

Step 5 Endpoint case $q = 2, r = \frac{2d}{d-2}$ **for** $d \geqslant 3$: See [Keel-Tao 1998]. □

Remark 2.3 (TT^* argument)**.** *We can rewrite the proof in a more elegant way. Let* $T : L^2(\mathbb{R}^d) \mapsto L^q(\mathbb{R}; L^r(\mathbb{R}^d))$ *be defined as*

$$(Tf)(t,x) = S(t)f(x),$$

then its formal adjoint $T^* : L^{q'}(\mathbb{R}; L^{r'}(\mathbb{R}^d)) \mapsto L^2(\mathbb{R}^d)$ *is defined as*

$$(T^*g)(x) = \int_{-\infty}^{\infty} S(-t)g(t,x)dt,$$

and their composition $TT^* : L^{q'}(\mathbb{R}; L^{r'}(\mathbb{R}^d)) \mapsto L^q(\mathbb{R}; L^r(\mathbb{R}^d))$ *reads as*

$$(TT^*g)(t,x) = \int_{-\infty}^{\infty} S(t-t')g(t',x)dt'.$$

Then the following a priori estimates are equivalent:

$$\|Tf\|_{L^q(\mathbb{R};L^r(\mathbb{R}^d))} \lesssim \|f\|_{L^2(\mathbb{R}^d)},$$

$$\|T^*g\|_{L^2(\mathbb{R}^d)} \lesssim \|g\|_{L^{q'}(\mathbb{R};L^{r'}(\mathbb{R}^d))},$$
$$\|TT^*g\|_{L^q(\mathbb{R};L^r(\mathbb{R}^d))} \lesssim \|g\|_{L^{q'}(\mathbb{R};L^{r'}(\mathbb{R}^d))},$$

where the last inequality for (q,r) admissible exponent pair with $2 < q < \infty$ is ensured by Step 1 above. Hence we deduce from the facts that

$$T : L^2 \mapsto L^q(\mathbb{R}; L^r(\mathbb{R}^d)), \quad T^* : L^{\tilde{q}'}(\mathbb{R}; L^{\tilde{r}'}(\mathbb{R}^d)) \mapsto L^2(\mathbb{R}^d),$$

are linear bounded operators that $TT^* : L^{\tilde{q}'}(\mathbb{R}; L^{\tilde{r}'}(\mathbb{R}^d)) \mapsto L^q(\mathbb{R}; L^r(\mathbb{R}^d))$ is linear bounded operator. By Christ-Kiselev's Lemma the truncated operator $\widetilde{TT^*}g = \int_0^t S(t-t')g(t')dt'$ also maps from $L^{\tilde{q}'}(\mathbb{R}; L^{\tilde{r}'}(\mathbb{R}^d))$ to $L^q(\mathbb{R}; L^r(\mathbb{R}^d))$ for any two admissible exponent pairs (q,r), $(\tilde{q},\tilde{r})$ with $q, \tilde{q} \in (2,\infty)$.

Remark 2.4 (Fourier restriction theorem)**.** *If we relabel $x = (x_1, \cdots, x_d, x_{d+1}) = (x,t) \in \mathbb{R}^{d+1}$ and $\widehat{u_0}$ is supported in the unit ball, then we can write*

$$\begin{aligned} S(t)u_0(x) &= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{i(x\cdot\xi - t|\xi|^2)} \widehat{u_0}(\xi)\, \mathrm{d}\xi \\ &= \frac{1}{(2\pi)^{d/2}} \int_M e^{i(x\cdot\xi + x_{d+1}\xi_{d+1})} \widehat{u_0}(\xi) d\mu, \end{aligned}$$

where M denotes the paraboloid in $\mathbb{R}^{d+1}$ with non-zero Gauss curvature:

$$M = \{(\xi, \xi_{d+1}) \,|\, \xi_{d+1} = -|\xi|^2\},$$

and the measure $d\mu = \psi_0\, \mathrm{d}\xi$, where ψ_0 is a C^∞ function of compact support taking value 1 on M and $|\xi| \leqslant 1$. Then the estimate

$$\|S(t)u_0\|_{L^{2(d+2)/d}(\mathbb{R}^{d+1})} \leqslant C\|u_0\|_{L^2(\mathbb{R}^d)}$$

follows also from the Fourier restriction theorem (Tomas-Stein Theorem).

2.2 L^2 theory

2.2.1 Local well-posedness in L^2

Theorem 2.3. *[LWP in L^2] Let p be an L^2-subcritical exponent, i.e. $1 < p < 1 + \frac{4}{d}$. Let $\kappa = \pm 1$. Let $u_0 \in L^2(\mathbb{R}^d)$.*
Then the Cauchy problem (NLS) *is locally well-posed LWP in $L^2(\mathbb{R}^d)$ in the following sense: There exist a positive time $T > 0$ depending on $\|u_0\|_{L^2(\mathbb{R}^d)}, p, d$, and a unique solution $u = u(t,x)$ defined on the time interval $[-T,T]$ such that*

$$u \in X_T := \left\{ u \in C([-T,T]; L^2(\mathbb{R}^d)) \,\middle|\, u \in L^q([-T,T]; L^{p+1}(\mathbb{R}^d)) \right\}$$

with admissible exponent pair $(q, p+1)$ *i.e.* $\frac{2}{q} + \frac{d}{p+1} = \frac{d}{2}$*(s.t.* $p+1 < 2 + \frac{4}{d} < q < \infty$*)*

and there exists a neighborhood U *of* u_0 *in* $L^2(\mathbb{R}^d)$ *such that*

$$\Phi : U \mapsto X_{T'}, \quad u_0 \mapsto u \text{ is Lipschitz continuous for any } T' < T.$$

Proof. We solve the integral equation (Duhamel) by searching for the fixed point of the mapping

$$\Psi : u \mapsto \Psi(u) = S(t)u_0 - i\kappa \int_0^t S(t-t')\big(|u(t')|^{p-1}u(t')\big)dt' \tag{2.40}$$

in the ball of the functional space X_T as

$$X_T(R) := \Big\{ u \in X_T \Big| \|u\|_{[-T,T]} := \|u\|_{L^\infty([-T,T];L^2(\mathbb{R}^d))} + \|u\|_{L^q([-T,T];L^{p+1}(\mathbb{R}^d))} \leqslant R \Big\}$$

with R, T to be determined later. We will use the Banach fixed-point theorem (contraction mapping theorem) in the complete metric space $(X_T(R), \|\cdot\|_{[-T,T]})$, and we shall prove that

- Ψ is a well-defined map in $X_T(R)$ with appropriately chosen R, T;
- Ψ is a contraction map in $X_T(R)$ for some small enough T.

Finally we conclude that there is a unique fixed point of Ψ and the flow map $\Phi : u_0 \mapsto u$ is Lipschitzian continuous from a neighborhood $U \subset L^2(\mathbb{R}^d)$ of u_0 to $X_T(R)$.
In the following C will denote some constant depending on p, d which may vary from line to line.
Step 1 Well-definedness of the map Ψ in $X_T(R)$
By Strichartz estimates in Theorem 2.2, we deduce that

$$\|S(t)u_0\|_{[-T,T]} \leqslant C(p,d)\|u_0\|_{L^2_x},$$

and

$$\begin{aligned}
\|\Psi(u) - S(t)u_0\|_{[-T,T]} &\leqslant C(p,d)\big\||u|^{p-1}u\big\|_{L^{q'}([-T,T];L^{(p+1)'}(\mathbb{R}^d))} \\
&\leqslant C\Big(\int_{-T}^{T} \big\||u|^{p-1}u\big\|^{q'}_{L^{\frac{p+1}{p}}(\mathbb{R}^d)} dt\Big)^{\frac{1}{q'}} \\
&= C\Big(\int_{-T}^{T} \|u\|^{pq'}_{L^{p+1}(\mathbb{R}^d)} dt\Big)^{\frac{1}{q'}}.
\end{aligned}$$

Since $1 < p < 1 + \frac{4}{d}$, we use Hölder's inequality $\|fg\|_{L^1} \leqslant \|f\|_{L^{\frac{q}{pq'}}} \|g\|_{L^{(1-\frac{pq'}{q})^{-1}}}$ to deduce

$$\begin{aligned}\|\Psi(u)\|_{[-T,T]} &\leqslant C\|u_0\|_{L^2} + C\Big(\int_{-T}^{T} \|u\|^q_{L^{p+1}(\mathbb{R}^d)}\, dt\Big)^{\frac{p}{q}} T^\theta \\ &\leqslant C\|u_0\|_{L^2} + C\|u\|^p_{[-T,T]} T^\theta,\end{aligned}$$

with $\theta = \frac{1}{q'}(1 - \frac{pq'}{q}) = \frac{1}{q'} - \frac{p}{q} = \frac{d}{4}(1 + \frac{4}{d} - p) > 0$.
We choose $R = 2C\|u_0\|_{L^2_x}$ and T_1 sufficiently small such that

$$\begin{aligned}&C(2C\|u_0\|_{L^2})^p(T_1)^\theta = C\|u_0\|_{L^2}, \text{ i.e. } T_1 = (2C)^{-\frac{p}{\theta}}\|u_0\|^\beta_{L^2_x} \\ &\text{with } \beta = \frac{1-p}{\theta} = \frac{4(1-p)}{d(1+\frac{4}{d}-p)} < 0,\end{aligned}$$

and hence for any $T \leqslant T_1$,

$$\text{if } \|u\|_{[-T,T]} \leqslant R, \text{ then } \|\Psi(u)\|_{[-T,T]} \leqslant C\|u_0\|_{L^2} + C\|u_0\|_{L^2} = R.$$

The continuity $\Phi(u) \in C([-T,T]; L^2_x(\mathbb{R}^d))$ follows from the continuity property of $S(t)u_0, \Psi(u) - S(t)u_0$ in $L^2(\mathbb{R}^d)$.
Step 2 Contraction map Ψ
Let $u, v \in X_T(R)$ and we calculate by Strichartz estimate

$$\begin{aligned}\|\Psi(u) - \Psi(v)\|_{[-T,T]} &= \Big\|\int_0^t S(t-t')\Big(|u(t')|^{p-1}u(t') - |v(t')|^{p-1}v(t')\Big)dt'\Big\|_{[-T,T]} \\ &\leqslant C\Big\||u|^{p-1}u - |v|^{p-1}v\Big\|_{L^{q'}([-T,T];L^{(p+1)'}(\mathbb{R}^d))}.\end{aligned}$$

Since $||u|^{p-1}u - |v|^{p-1}v| \leqslant C_1(|u|^{p-1} + |v|^{p-1})|u-v|$ for some constant C_1, we proceed as in Step 1 to obtain

$$\begin{aligned}&\|\Psi(u) - \Psi(v)\|_{[-T,T]} \leqslant CC_1\Big(\int_{-T}^{T}\big(\|u\|^{(p-1)q'}_{L^{p+1}_x} + \|v\|^{(p-1)q'}_{L^{p+1}_x}\big)\|u-v\|^{q'}_{L^{p+1}_x} dt\Big)^{\frac{1}{q'}} \\ &\leqslant CC_1\Big(\|u\|^{p-1}_{L^q([-T,T];L^{p+1}_x)} + \|v\|^{p-1}_{L^q([-T,T];L^{p+1}_x)}\Big)\|u-v\|_{L^q([-T,T];L^{p+1}_x)}T^\theta \\ &\leqslant C_2R^{p-1}T^\theta\|u-v\|_{[-T,T]} \text{ for some constant } C_2 \geqslant C.\end{aligned}$$

Hence we take T such that

$$C_2R^{p-1}T^\theta = \frac{1}{2} \text{ i.e. } T = (2C_2)^{-\frac{1}{\theta}}(2C)^{-\frac{p-1}{\theta}}\|u_0\|^\beta_{L^2_x} \leqslant T_1,$$

and the map Ψ is a contraction map on $X_T(R)$.

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Step 3 Conclusion
By Banach fixed point theorem, there exists a unique fixed point $u \in X_T(R)$ of the map Ψ and hence $u \in X_T(R)$ solves uniquely (NLS) with $R = C_3\|u_0\|_{L^2_x}$, $T = C_3^{-1}\|u_0\|^{\beta}_{L^2_x}$ for some large enough constant C_3 (to be determined later). Without loss of generality we can assume that for any initial data in the neighborhood of u_0: $U = \{v_0 \in L^2(\mathbb{R}^d) | \|u_0 - v_0\|_{L^2(\mathbb{R}^d)} < \|u_0\|_{L^2(\mathbb{R}^d)}\}$ such that $\|v_0\|_{L^2(\mathbb{R}^d)} < 2\|u_0\|_{L^2(\mathbb{R}^d)}$, there is a unique solution $v \in X_T(R)$ of (NLS).
We are going to show the Lipschitz continuity of the flow map $\Phi : U \mapsto X_{T'}(R)$ via $u_0 \mapsto u$ for all $T' < T$. Let $u_0, v_0 \in U$ and we calculate

$$\begin{aligned}\Phi(u_0) - \Phi(v_0) &= S(t)(u_0 - v_0) \\ &\quad - i\kappa \int_0^t S(t-t')\Big(|\Phi(u_0)(t')|^{p-1}\Phi(u_0)(t') - |\Phi(v_0)(t')|^{p-1}\Phi(v_0)(t')\Big)dt'.\end{aligned}$$

As in Step 2, we derive that

$$\begin{aligned}\|\Phi(u_0) - \Phi(v_0)\|_{[0,T]} &\leqslant C\|u_0 - v_0\|_{L^2_x} \\ &\quad + C\big(\|\Phi(u_0)\|^{p-1}_{[0,T]} + \|\Phi(v_0)\|^{p-1}_{[0,T]}\big)\|\Phi(u_0) - \Phi(v_0)\|_{[0,T]}T^{\theta} \\ &\leqslant C\|u_0 - v_0\|_{L^2_x} + C_2\big(\|u_0\|^{p-1}_{L^2_x} + \|v_0\|^{p-1}_{L^2_x}\big)\|\Phi(u_0) - \Phi(v_0)\|_{[0,T]}T^{\theta},\end{aligned}$$

such that

$$\|\Phi(u_0) - \Phi(v_0)\|_{[0,T]} \leqslant 2C_3\|u_0 - v_0\|_{L^2_x},$$

if $T = C_3^{-1}\|u_0\|^{\beta}_{L^2_x}$ for sufficiently large C_3. □

Remark 2.5. *The nonlinear Schrödinger equation* (NLS) *holds in the distribution sense: It follows from the Duhamel formulation* (Duhamel) *and the well-definedness of the nonlinearity when $u \in X_T$*

$$|u|^{p-1}u \in L^{\frac{q}{p}}([-T,T]; L^{\frac{p+1}{p}}(\mathbb{R}^d)), \quad \frac{q}{p} > q' \text{ if } p < 1 + \frac{4}{d}.$$

Then by Strichartz estimates the solution u itself belongs to any functional space $L^{\tilde{q}}([-T,T]; L^{\tilde{r}}(\mathbb{R}^d))$ for any admissible exponent pair $(\tilde{q}, \tilde{r})$.
By Sobolev embedding $H^1(\mathbb{R}^d) \hookrightarrow L^{p+1}(\mathbb{R}^d) = \big(L^{\frac{p+1}{p}}(\mathbb{R}^d)\big)'$, $|u|^{p-1}u \in L^{\frac{q}{p}}([-T;T]; H^{-1}(\mathbb{R}^d))$ and hence the equation (NLS) *makes sense at least in $L^{q'}([-T,T]; H^{-2}(\mathbb{R}^d))$. However it is in generally not true that we can take the L^2_x-inner product between the equation* (NLS) *and the solution u directly to show the mass conservation law: That is why we first do regularization and then take the L^2_x inner product in next Subsection 2.2.2.*

2.2.2 Global well-posedness in L^2

Theorem 2.4. *[GWP in L^2] Let $1 < p < 1 + \frac{4}{d}$. The solution obtained in Theorem 2.3 exists globally in time such that*

$$u \in C(\mathbb{R}; L^2_x) \cap L^q_{\text{loc}}(\mathbb{R}; L^{p+1}_x) \text{ and } \|u(t,\cdot)\|_{L^2_x} = \|u_0\|_{L^2_x}, \forall t \in \mathbb{R}. \tag{2.41}$$

Proof. We show the conservation of the L^2_x-norm, i.e. the mass conservation law (1.9), rigorously for $u \in X_T$ satisfying (NLS).

Step 1 Regularization

Take $\varphi \in C_0^\infty(\mathbb{R}^d)$ with $\varphi \geqslant 0$, $\int_{\mathbb{R}^d} \varphi(x)\,\mathrm{d}x = 1$. Denote $\varphi_n(x) = n^d \varphi(nx)$. Similarly we take $\psi \in C_0^\infty([-T,T])$, $\psi \geqslant 0$, $\int_{\mathbb{R}} \psi(t)dt = 1$, $\operatorname{Supp}(\psi) \subset [-\frac{T}{2}, \frac{T}{2}]$, and denote $\psi_m(t) = m\psi(mt)$, $m \geqslant N$ with N sufficiently large. Since $u \in C([-T,T]; L^2_x) \cap L^q([-T,T]; L^{p+1}_x)$, we have

$$\psi_m *_t \varphi_n *_x u \to u \text{ in } C\big(I_N; L^2_x\big) \cap L^q\big(I_N; L^{p+1}_x\big),$$
$$\psi_m *_t \varphi_n *_x (|u|^{p-1}u) \to (|u|^{p-1}u) \text{ in } L^{q'}(I_N; L_x^{(p+1)'})$$

as $m, n \to \infty$. Here we denote $I_N = (-(1-\frac{1}{2N})T, (1-\frac{1}{2N})T)$.
We take the convolution of (NLS) with φ_n and then with ψ_m to arrive at

$$i\partial_t u_{m,n} + \Delta u_{m,n} = \kappa \psi_m *_t \varphi_n *_x (|u|^{p-1}u), \quad u_{m,n} = \psi_m *_t \varphi_n *_x u. \tag{2.42}$$

We test the above equation for $u_{m,n}$ by $\overline{u_{m,n}} \in \mathcal{S}(I_N \times \mathbb{R}^d_x)$ and then take the imaginary part. Similarly as the derivation of (1.9), we derive

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d} |u_{m,n}|^2\,\mathrm{d}x = \kappa \mathrm{Im} \int_{\mathbb{R}^d} \big(\psi_m * \varphi_n * (|u|^{p-1}u)\big)\overline{u_{m,n}}\,\mathrm{d}x,$$

for all $t \in I_N$.

Step 2 Pass to the limit

For any $T' \in I_N$, we derive from the above equality that

$$\begin{aligned}\frac{1}{2}(\|u_{m,n}(T')\|_{L^2_x} - \|u_{m,n}(0)\|_{L^2_x}) &= \kappa \mathrm{Im} \int_{[0,T']\times\mathbb{R}^d} \big(\psi_m * \varphi_n * (|u|^{p-1}u)\big)\overline{u_{m,n}}\,\mathrm{d}x\,dt \\ &\to \kappa \mathrm{Im} \int_{[0,T']\times\mathbb{R}^d} (|u|^{p-1}u)\overline{u}\,\mathrm{d}x\,dt = 0,\end{aligned}$$

and hence

$$\|u(T')\|_{L^2_x} = \lim_{m,n\to\infty} \|u_{m,n}(T')\|_{L^2_x} = \lim_{m,n\to\infty} \|u_{m,n}(0)\|_{L^2_x} = \|u_0\|_{L^2_x}.$$

As the above holds for all $T' \in I_N$, it indeed holds for all $T' \in [-T,T]$.
Recall that the existence time T depends only on $p, d, \|u_0\|_{L^2_x}$, the solution obtained in Theorem 2.3 can be extended to all the time by uniqueness continuation. □

2.2.3 L^2 critical case

Let us consider the L^2 critical case $p = 1 + \frac{4}{d}$, which is quite interesting case: Recall the cubic nonlinear Schrödinger equations (1.28) and (1.33) with $(p, d) = (3, 2)$. In this case, the admissible pair

$$(q, p+1) = (2 + \frac{4}{d}, 2 + \frac{4}{d}) = (p+1, p+1).$$

In this critical case, we can still get the local-in-time well-posedness result by taking the existence time T_0 sufficiently small such that $\|S(t)u_0\|_{L^{p+1}([-T_0,T_0]\times\mathbb{R}^d)}$ is small on this small time interval. However, such choice of T_0 depends indeed on u_0 itself (not only on $\|u_0\|_{L^2}$), and hence the mass conservation law can not imply immediately the global-in-time wellposedness. Nevertheless, we can assume a priori the small size of the initial data, such that the contraction map argument still works and gives the global-in-time result.

Theorem 2.5 (LWP & GWP for L^2 critical case: A preliminary version)**.** *Let $p = 1 + \frac{4}{d}$ be the $L^2(\mathbb{R}^d)$ critical exponent. Let $\kappa = \pm 1$. Then*

- (NLS) *is locally well-posed in $L^2(\mathbb{R}^d)$ such that for any $u_0 \in L^2(\mathbb{R}^d)$, there exists a unique solution*

  $$u \in X_T = \{u \in C([-T, T]; L^2(\mathbb{R}^d)) \mid u \in L^{p+1}_{t,x}([-T, T] \times \mathbb{R}^d)\},$$

  where $T > 0$ depending on u_0, d and there exists a neighborhood U of u_0 such that the flow map $\Phi : L^2 \mapsto X_T$ via $\Phi : u_0 \mapsto u$ is Lipschitz continuous;

- *There exists a sufficiently small constant $\varepsilon_0 > 0$ depending on d such that if $\|u_0\|_{L^2_x} \leqslant \varepsilon_0$ then* (NLS) *is globally well-posed in $L^2(\mathbb{R}^d)$ and the unique solution belongs to*

  $$C(\mathbb{R}; L^2_x(\mathbb{R}^d)) \cap L^{p+1}_{t,x}(\mathbb{R} \times \mathbb{R}^d).$$

Proof. (**Exercise**.)

Step 1 Smallness of $\|S(t)u_0\|_{L^{p+1}([-T_0,T_0]\times\mathbb{R}^d)}$ for small T_0
For any $\varepsilon > 0$, for any $u_0 \in L^2$, there exists a neighborhood U of u_0 and $T_0 > 0$ such that

$$\|S(t)v_0\|_{L^{p+1}_{t,x}([-T_0,T_0]\times\mathbb{R}^d)} \leqslant \varepsilon, \quad \forall v_0 \in U.$$

Indeed, the mapping $S(t) : L^2(\mathbb{R}^d) \mapsto X_T$ is locally Lipschitz. Therefore we can take the neighborhood of u_0 as $U = \{v_0 \in L^2_x \mid \|v_0 - u_0\|_{L^2_x} < C^{-1}\varepsilon\}$ for sufficiently large C, and hence it remains to show the above for u_0. It follows by dominated convergence theorem or approximation argument.

Step 2 LWP in L^2

We prove the local well-posedness result by searching for the fixed point for the mapping Ψ defined in (2.40) in

$$\{u \in X_{T_0} \mid \|u\|_{L_t^\infty([-T_0,T_0];L_x^2(\mathbb{R}^d))} \leqslant 2C\|u_0\|_{L_x^2},\ \|u\|_{L_{t,x}^{p+1}([-T_0,T_0]\times\mathbb{R}^d)} \leqslant 2\varepsilon\},$$

for some sufficiently small ε depending on $\|u_0\|_{L_x^2}, d$ and T_0 depending on ε, u_0. By Step 1, we can assume that there exists T_0 such that for any initial data v_0 in the neighborhood U there exists a unique solution $v \in X_{T_0}$ with $\|v\|_{L_{t,x}^{p+1}([-T_0,T_0]\times\mathbb{R}^d)} \leqslant 2\varepsilon$.

We prove that the flow map $\Phi : U \mapsto X_{T_0}$ is Lipschitz continuous.

Step 3 Small initial data case

We prove the global well-posedness result in L^2 for small initial data $\|u_0\|_{L_x^2} \leqslant \varepsilon_0$, similarly as in Step 2. □

[20.05.2022]

[23.05.2022]

Remark 2.6 (Blow-up criterion)**.** *If I is the maximal time interval of the existence of the solution u of* (NLS) *for $p = 1 + \frac{4}{d}$ (in the sense of Theorem 2.5) with* $\sup(I) < \infty$*, then*

$$\lim_{T\nearrow\sup(I)} \|u\|_{L^{2+\frac{4}{d}}([0,T]\times\mathbb{R}^d)} = \infty. \tag{2.43}$$

The corresponding result holds true for I with $\inf(I) > -\infty$*.*
Indeed, let $t_0 \in I$ be any time, then there exists $T(t_0)$ such that

$$\|u\|_{L^{2+\frac{4}{d}}([t_0-T(t_0),t_0+T(t_0)]\times\mathbb{R}^d)} \leqslant \varepsilon_0.$$

Hence for any compact interval $J \subset I$,

$$\|u\|_{L^{2+\frac{4}{d}}(J\times\mathbb{R}^d)} < \infty.$$

If (2.43) *does not hold:* $\lim_{T\nearrow\sup(I)} \|u\|_{L^{2+\frac{4}{d}}([0,T]\times\mathbb{R}^d)} < \infty$*, then $u(t)$ converges in L^2 as $t \nearrow$* $\sup(I)$*, which contradicts to the maximality of the existence time interval I.*

Relying on heavy analysis techniques such as profile decomposition, concentration-compactness method and interaction Morawetz estiamtes, Dodson 2016 proved the global-in-time wellposedness (and scattering) result for the defocusing mass-critical case, without smallness assumption on the initial mass size. See Dodson's book ,,Defocusing nonlinear Schrödigner equations" (Cambridge university press, 2019) for more details.

Theorem 2.6 (Dodson's GWP for L^2-critical case)**.** *Let $p = 1 + \frac{4}{d}$, $\kappa = 1$, $d \geqslant 1$. Then the Cauchy problem* (NLS) *is globally-in-time well-posed in $L^2(\mathbb{R}^d)$ in the sense in Theorem 2.3.*

Very brief sketch of the proof. For any initial size $m \in (0, \infty)$, we define the scattering size function

$$A(m) = \sup \Big\{ \|u\|_{L^{2+\frac{4}{d}}_{t,x}(\mathbb{R} \times \mathbb{R}^d)} \Big| u \text{ solves } i\partial_t u + \Delta u = |u|^{\frac{4}{d}} u \\ \text{with } u|_{t=0} = u_0(x), \quad \|u_0\|_{L^2(\mathbb{R}^d)} = m \Big\}.$$

If we can show $A(m) < \infty$ for any $m < \infty$, then by Remark 2.6 the GWP (and the scattering) results hold true.
One argues by condradiction argument. Firstly, by Theorem 2.5, $A(m) \lesssim m$ if $m \leqslant \varepsilon_0$. On the other side, one can show that $A(m)$ is continuous and increasing function, such that

$$\{m : A(m) = \infty\} = [m_0, \infty) \text{ for some } m_0 \in (0, \infty].$$

We assume by contraction that $m_0 < \infty$. Let $u_n(0)$ be a sequence of initial data such that

$$A(\|u_n(0)\|_{L^2}) \nearrow +\infty, \quad \|u_n(0)\|_{L^2} \nearrow m_0.$$

Let u_n be the corresponding solutions such that

$$\|u_n\|_{L^{2+\frac{4}{d}}_{t,x}([0,\infty)\times\mathbb{R}^d)} \nearrow +\infty, \quad \|u_n\|_{L^{2+\frac{4}{d}}_{t,x}((-\infty,0]\times\mathbb{R}^d)} \nearrow +\infty. \tag{2.44}$$

Step 1. Profile decomposition. For any $j \in [1, \infty)$, there exist $\phi^j \in L^2(\mathbb{R}^d)$ (ϕ^j can be zero), $(t^j_n, x^j_n) \in \mathbb{R} \times \mathbb{R}^d$, $\xi^j_n \in \mathbb{R}^d$, $\lambda^j_n \in (0, \infty)$, such that

$$u_n(0) = \sum_{j=1}^{J} e^{-it^j_n \Delta} e^{ix\cdot\xi^j_n} \frac{1}{(\lambda^j_n)^{\frac{d}{2}}} \phi^j(\frac{x - x^j_n}{\lambda^j_n}) + w^J_n, \tag{2.45}$$

with

$$\lim_{J\to\infty} \limsup_{n\to\infty} \|e^{it\Delta} w^J_n\|_{L^{2+\frac{4}{d}}_{t,x}} = 0,$$

$$(\lambda_n^j)^{\frac{d}{2}} e^{-i\xi_n^j\cdot(\lambda_n^j x + x_n^j)}\Big(e^{it_n^j\Delta} w_n^J\Big)(\lambda_n^j x + x_n^j) \rightharpoonup 0,$$

$$\lim_{n\to\infty}\Big(\|u_n(0)\|_{L^2}^2 - \sum_{j=1}^J \|\phi^j\|_{L^2}^2 - \|w_n^J\|_{L^2}^2\Big) = 0,$$

$$\frac{\lambda_n^j}{\lambda_n^k} + \frac{\lambda_n^k}{\lambda_n^j} + \lambda_n^k|\xi_n^j - \xi_n^k| + \frac{|t_n^j - t_n^k|}{(\lambda_n^j)^2} + \frac{|x_n^j - x_n^k + 2(t_n^k - t_n^j)\xi_n^j|}{\lambda_n^k} \to \infty, \qquad \text{for } j \neq k.$$

Furthermore, for any j, t_n^j is uniformly bounded and converges to some $t^j \in \mathbb{R}$. Otherwise, if e.g. $t_n^1 \nearrow +\infty$, then

$$u_n(0) = e^{-it_n^1\Delta} e^{ix\cdot\xi_n^1} \frac{1}{(\lambda_n^1)^{\frac{d}{2}}} \phi^1(\frac{x - x_n^1}{\lambda_n^1}) + w_n^1,$$

with $\lim_{n\to\infty}\|w_n^1\|_{L^2} \leqslant m_0 - \varepsilon < m_0$, and hence $\lim_{n\to\infty} A(\|w_n^1\|_{L^2}) < \infty$. (By the long-time perturbation theorem), this implies contradiction between (2.44) and the following fact (resulting from dominated convergence and Strichartz estimates)

$$\lim_{n\to\infty} \|e^{it\Delta}(u_n(0) - w_n^1)\|_{L_{t,x}^{2+\frac{4}{d}}([0,\infty)\times\mathbb{R}^d)} = 0.$$

Thus we may assume $t_n^j \to t^j \in \mathbb{R}$, and replace ϕ^j by $e^{it^j\Delta}\phi^j$ (without loss of generality),

$$u_n(0) = \sum_{j=1}^J e^{ix\cdot\xi_n^j} \frac{1}{(\lambda_n^j)^{\frac{d}{2}}} \phi^j(\frac{x - x_n^j}{\lambda_n^j}) + w_n^J.$$

Step 2. Mass-critical almost-periodicity. We claim that indeed $\|\phi^1\|_{L^2} = m_0$, $\phi^j = 0$ for all $j \geqslant 2$, and $w_n^1 \to 0$ strongly in $L^2(\mathbb{R}^d)$. And hence $u_n(0)$ (modulo translation, scaling and modulation symmetries) converges to $u_0 \in L^2(\mathbb{R}^d)$ with $A(\|u_0\|_{L^2}) = \infty$, such that the solution u with the initial data u_0 satisfies

$$\|u\|_{L_{t,x}^{2+\frac{4}{d}}(I\cap[0,\infty)\times\mathbb{R}^d)} = \|u\|_{L_{t,x}^{2+\frac{4}{d}}(I\cap(-\infty,0]\times\mathbb{R}^d)} \nearrow +\infty, \tag{2.46}$$

for the maximal time interval I. Let $t_n \in I$, and $\tilde{u}_n(0) = u(t_n)$, then the same argument implies $u(t_n)$ (modulo the symmetries) converges in $L^2(\mathbb{R}^d)$. Hence $u(t)$ lies in GK for all $t \in I$, where $K \subset L^2$ is a compact set, and G denotes the symmetry group. The Arzelà-Ascoli theorem implies the existence of the mass-critical almost-periodic solution $u(t,x)$, $t \in I$, $x \in \mathbb{R}^d$, such that for all $\eta > 0$,

$$\int_{|x-x(t)|\geqslant\frac{C(\eta)}{N(t)}} |u(t,x)|^2\,\mathrm{d}x + \int_{|\xi-\xi(t)|\geqslant C(\eta)N(t)} |\hat{u}(t,\xi)|^2\,\mathrm{d}\xi < \eta,$$

for some time-dependent functions $x(t), \xi(t), N(t)$.
Indeed, let v^j be the solution with the initial data ϕ^j. We assume by contradiction that $\sup_j \|\phi^j\|_{L^2} \leqslant m_0 - \varepsilon$, such that $\|v^j\|_{L^{2+\frac{4}{d}}_{t,x}} \leqslant A(m_0-\varepsilon) < \infty$. For the j such that $\|\phi^j\|_{L^2} \leqslant \varepsilon_0$, we know $\|v^j\|_{L^{2+\frac{4}{d}}_{t,x}} \lesssim \|\phi\|_{L^2}$. As there are only finitely many j such that $\|\phi^j\|_{L^2} > \varepsilon_0$, we can conclude

$$\sup_J \lim_{n\to\infty} A\Big(\Big\|\sum_{j=1}^J e^{ix\cdot\xi_n^j}\frac{1}{(\lambda_n^j)^{\frac{d}{2}}}\phi^j(\frac{x-x_n^j}{\lambda_n^j})\Big\|_{L^2}\Big) < \infty,$$

which contradicts (2.44).
Step 3. Trivial almost periodic solutions. We claim that there are only trivial almost-periodic solution which contradicts (2.46). And hence we can conclude finally $A(m) < \infty$ for all $m < \infty$.
Indeed, if there is a nonzero almost-periodic solution, then for at least one such solution, one of the following two scenarios hold:

- $I = \mathbb{R}$, $N(t) \leqslant 1$, $\forall t \in \mathbb{R}$. It is convenient to split into two scenarios:
  - Soliton-like solution: $\int_0^\infty N(t)^3 dt = \infty$;
  - Rapid cascade solution: $\int_0^\infty N(t)^3 dt < \infty$.
- self-similar solution: $I = (0,\infty)$, $N(t) = t^{-\frac{1}{2}}$.

One can show the the self-similar solution indeed has $\dot{H}^1$-regularity. By the H^1-GWP result below, the solution can not blow up at 0, which contradicts with $I = (0,\infty)$. Similar arguments work for the case of rapid cascade.
As for the soliton-like case $I = \mathbb{R}$, $N(t) \leqslant 1$, with $\int_0^\infty N(t)^3 dt = \infty$, one can estimate the solution using a frequency-localized interaction Morawetz estimate (indeed localized in the low-frequencies due to the low regularity assumption $u_0 \in L^2$). The error terms due to this truncation can be estimated by long-time Strichartz estimates. More precisely, for $d \geqslant 3$, one can define the long-time Strichartz seminorm

$$\|u\|^2_{X([a,b]\times\mathbb{R}^d)} = \sup_{0\leqslant j\leqslant k_0} 2^{j-k_0}\|e^{ix\cdot\xi(t)}P_{\geqslant j}(e^{-ix\cdot\xi(t)}u)\|^2_{L^2_t L^{\frac{2d}{d-2}}([a,b]\times\mathbb{R}^d)}$$

□

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Remark 2.7.

- *We notice that in Theorem 2.5 there are well-posedness results in $L^2(\mathbb{R}^d)$ for the L^2-critical case if there are smallness conditions, either on the existing time or on the size of the initial data.*

- *In the defocusing L^2 critical case, there were constant contributions to global well-posedness results for large initial data in the literature (before Dodson):*
  - under an additional decay assumption $|x|^m u_0 \in L^2(\mathbb{R}^d)$, $m > 3/5$, see [Bourgain 1998 JAM];
  - under an additional regularity assumption $u_0 \in H^s(\mathbb{R}^d)$, $s > 4/7$, see [Colliander-Keel-Staffilani-Takaoka-Tao 2008 DCDS-A];
  - in the radial case, see [Tao-Visan-Zhang 2007 DMJ], [Killip-Tao-Visan 2009 JEMS] for higher and two dimensional cases respectively.

  We are going to show the global well-posedness result in H^1_x for the defocusing H^1-subcritical case which includes the L^2-critical case.

- *In the focusing L^2 critical case, [Dodson 2015] showed the global well-posedness result with the initial mass below the soliton solution threshold: $\|u_0\|_{L^2} < \|Q\|_{L^2}$ where Q is the solution of the elliptic equation* (1.22). *There may be blowup phenomena for $\|u_0\|_{L^2} \geqslant \|Q\|_{L^2}$.*

  We are going to consider these long time behaviors in Section 3, but with regular initial data $u_0 \in \Sigma(\mathbb{R}^d)$.

- *In the supercritical case $p > 1 + \frac{4}{d}$, there are ill-posedness results for* (NLS)*, see [Christ-Colliander-Tao 2003 arXiv]: If $s_c = \frac{d}{2} - \frac{2}{p-1} > 0$, then for any $s < s_c$, for any $0 < \delta, \epsilon < 1$ and any $t > 0$, there exist solutions u_1, u_2 of* (NLS) *with smooth initial data $u_1(0), u_2(0) \in \mathcal{S}$ such that*

  $$\|u_1(0)\|_{H^s} + \|u_2(0)\|_{H^s} \leqslant C\epsilon, \quad \|u_1(0) - u_2(0)\|_{H^s} \leqslant C\delta,$$
  $$\|u_1(t) - u_2(t)\|_{H^s} \geqslant c\epsilon.$$

  In the focusing case, the blowup phenominon in finite time from smooth data can be proved simply via the virial identity and we can construct the blowup example by applying scaling and Galilean transformation to the soliton solutions.

2.3 Sobolev spaces

2.3.1 Sobolev spaces $H^s(\mathbb{R}^d)$

Recall the definition (1.6) of the Sobolev space $H^s(\mathbb{R}^d)$, $s \in \mathbb{R}$ as follows

$$H^s(\mathbb{R}^d) := \{f \in \mathcal{S}'(\mathbb{R}^d) \mid \|f\|^2_{H^s(\mathbb{R}^d)} := \int_{\mathbb{R}^d} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 \, \mathrm{d}\xi \ < \infty\}. \quad (2.47)$$

If $s \in \mathbb{N}$, then

$$H^s(\mathbb{R}^d) = \{f \in L^2(\mathbb{R}^d) \mid \partial^\alpha f \in L^2(\mathbb{R}^d), \forall \text{multi-index } \alpha \text{ with } |\alpha| \leqslant s\}.$$

The Sobolev spaces have the following properties:

- $H^{s_1}(\mathbb{R}^d) \subset H^{s_0}(\mathbb{R}^d)$ if $s_0 \leqslant s_1$.
- The following interpolation inequality holds by Hölder's inequality:

$$\|f\|_{H^{s_\theta}} \leqslant \|f\|^{1-\theta}_{H^{s_0}} \|f\|^{\theta}_{H^{s_1}}, \text{ with } s_\theta = (1-\theta)s_0 + \theta s_1. \quad (2.48)$$

- $H^s(\mathbb{R}^d)$ is a Hilbert space with the inner product

$$(u, v)_{H^s(\mathbb{R}^d)} = \int_{\mathbb{R}^d} (1 + |\xi|^2)^s \hat{u}(\xi) \overline{\hat{v}(\xi)} \, \mathrm{d}\xi \, .$$

  and it is isometrically anti-isomorphic to its dual space $(H^s(\mathbb{R}^d))'$.

- $H^{-s}(\mathbb{R}^d)$ can be identified as the set of the continuous linear functionals on $H^s(\mathbb{R}^d)$ via $L^2(\mathbb{R}^d)$-inner product: Let $f \in \mathcal{S}'(\mathbb{R}^d)$ and $\hat{f} \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$, then

$$f \in H^{-s}(\mathbb{R}^d) \Leftrightarrow \sup{}_{g \in \mathcal{S}, \|g\|_{H^s} \leqslant 1} |\langle f, g \rangle_{\mathcal{S}', \mathcal{S}}| = \sup{}_{g \in \mathcal{S}, \|g\|_{H^s} \leqslant 1} \Big| \int_{\mathbb{R}} f \bar{g} \, \mathrm{d}x \Big| < \infty,$$

  and we will denote $\langle f, g \rangle_{H^{-s}, H^s} = \langle (1 + |\xi|^2)^{-s/2} \hat{f}, (1 + |\xi|^2)^{s/2} \hat{g} \rangle_{L^2}$.

Theorem 2.7. *[Sobolev embedding for $H^s(\mathbb{R}^d)$] The following Sobolev embedding results hold true:*

- *If $0 \leqslant s < \frac{d}{2}$, then $H^s(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d)$ for any $p \in [2, p_c]$ with $\frac{d}{2} - s = \frac{d}{p_c}$ continuously and there exists a constant C depending on d, s such that*

$$\|f\|_{L^{p_c}(\mathbb{R}^d)} \leqslant C \|f\|_{\dot{H}^s(\mathbb{R}^d)}, \quad \forall f \in \mathcal{D}(\mathbb{R}^d), \quad (2.49)$$

  where the homogeneous Sobolev seminorm is defined in (1.7)*: $\|f\|^2_{\dot{H}^s(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |\xi|^{2s} |\hat{f}(\xi)|^2 \, \mathrm{d}\xi$.*

- *If* $s = \frac{d}{2}$, *then* $H^s(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d)$, *for all* $2 \leqslant p < \infty$ *continuously;*
- *If* $s > \frac{d}{2}$, *then* $H^s(\mathbb{R}^d) \hookrightarrow C_0(\mathbb{R}^d)$ *continuously;*
- *If* $0 \leqslant s < \frac{d}{2}$, *then* $L^{p'}(\mathbb{R}^d) \hookrightarrow H^{-s}(\mathbb{R}^d)$, $p \in [2, p_c]$ *i.e.* $p' \in [(\frac{1}{2} + \frac{s}{d})^{-1}, 2]$ *continuously.*

Proof. **Step 1 Proof of** (2.49)
The case $s = 0$ is obvious and we consider the case $0 < s < \frac{d}{2}$, $p_c = \frac{2d}{d-2s} > 2$. For any $A > 0$ we can decompose f into low- and high- frequency parts as follows:

$$f = f_l + f_h, \quad \hat{f}_l = \mathbf{1}_{<A}\hat{f}, \quad \hat{f}_h = \mathbf{1}_{\geqslant A}\hat{f}.$$

Then we can control the low frequency part f_l by

$$\begin{aligned}\|f_l\|_{L^\infty} &\leqslant (2\pi)^{-\frac{d}{2}}\|\hat{f}_l\|_{L^1} = (2\pi)^{-\frac{d}{2}}\int_{|\xi|<A} |\hat{f}(\xi)||\xi|^s|\xi|^{-s}\,\mathrm{d}\xi \\ &\leqslant (2\pi)^{-\frac{d}{2}}\big\||\hat{f}(\xi)||\xi|^s\big\|_{L^2}\Big(\int_{|\xi|<A}|\xi|^{-2s}\,\mathrm{d}\xi\Big)^{\frac{1}{2}} \\ &\leqslant (2\pi)^{-\frac{d}{2}}\Big(\frac{\omega_d}{d-2s}\Big)^{\frac{1}{2}}\|f\|_{\dot{H}^s}A^{\frac{d}{2}-s} =: C\|f\|_{\dot{H}^s}A^{\frac{d}{p_c}}.\end{aligned}$$

We write

$$\|f\|_{L^p}^p = p\int_0^\infty \lambda^{p-1}|\{|f| \geqslant \lambda\}|d\lambda,$$

and we are going to estimate $|\{x \in \mathbb{R}^d \mid |f(x)| \geqslant \lambda\}|$ for each $\lambda \in (0, \infty)$. Indeed, for any $\lambda > 0$, we take $A = A(\lambda) = (4^{-1}C^{-1}\|f\|_{\dot{H}^s}^{-1}\lambda)^{\frac{p_c}{d}}$ such that the low frequency part $\|f_l\|_{L^\infty} \leqslant \lambda/4$ and hence

$$\begin{aligned}|\{x \in \mathbb{R}^d \mid |f(x)| \geqslant \lambda\}| &\leqslant |\{x \in \mathbb{R}^d \mid |f_h(x)| \geqslant \lambda/2\}| \leqslant 4\lambda^{-2}\|f_h\|_{L^2}^2 = 4\lambda^{-2}\|\widehat{f}_h\|_{L^2}^2 \\ &= 4\lambda^{-2}\int_{\left\{\xi\in\mathbb{R}^d \,\mid\, 4C\|f\|_{\dot{H}^s}|\xi|^{\frac{d}{p_c}}\geqslant\lambda\right\}} |\hat{f}|^2\,\mathrm{d}\xi\,.\end{aligned}$$

Therefore

$$\begin{aligned}\|f\|_{L^{p_c}}^{p_c} &= p_c\int_0^\infty \lambda^{p_c-1}|\{|f| \geqslant \lambda\}|d\lambda \\ &\leqslant 4p_c\int_{\left\{\xi \,\mid\, 4C\|f\|_{\dot{H}^s}|\xi|^{\frac{d}{p_c}}\geqslant\lambda\right\}} \lambda^{p_c-3}|\hat{f}|^2\,\mathrm{d}\xi\,d\lambda\end{aligned}$$

$$\leqslant \frac{4p_c}{p_c-2}\int_{\mathbb{R}^d}(4C\|f\|_{\dot H^s}|\xi|^{\frac{d}{p_c}})^{(p_c-2)}|\hat f(\xi)|^2\,\mathrm{d}\xi \;\leqslant \frac{2d}{s}(4C)^{p_c-2}\|f\|_{\dot H^s}^{p_c}.$$

Step 2 Case $0\leqslant s<\frac{d}{2}$
By interpolation of Lebesgue spaces and $\|f\|_{H^s}\sim\|f\|_{L^2}+\|f\|_{\dot H^s}$, we have the Sobolev embedding $H^s\hookrightarrow L^p$, $\forall p\in[2,p_c]$.
Step 3 Case $s=\frac{d}{2}$
For any $p\in[2,\infty)$, there exists $s_0=\frac{d}{2}-\frac{d}{p}\in[0,\frac{d}{2})$ such that $H^{\frac{d}{2}}\hookrightarrow H^{s_0}\hookrightarrow L^p$.
Step 4 Case $s>\frac{d}{2}$
Since

$$\begin{aligned}\|\hat f\|_{L^1(\mathbb{R}^d)} &= \int_{\mathbb{R}^d}(1+|\xi|^2)^{s/2}|\hat f(\xi)|(1+|\xi|^2)^{-s/2}\,\mathrm{d}\xi\\ &\leqslant \|f\|_{H^s}\big\|(1+|\xi|^2)^{-s/2}\big\|_{L^2(\mathbb{R}^d)}\leqslant C\|f\|_{H^s},\ \text{if } s>\frac{d}{2},\end{aligned}$$

the function f as the inverse Fourier transform of a L^1-function is bounded, continuous and tends to 0 at infinity by Riemann-Lebesgue Lemma.
Step 5 Case $-s\in(-\frac{d}{2},0]$
By density, it suffices to show $\|f\|_{H^{-s}}\leqslant C\|f\|_{L^{p'}}$, $0\leqslant\frac{d}{p'}-\frac{d}{2}\leqslant s$ for $f\in\mathcal{S}$. Indeed, since $\frac{d}{2}-\frac{d}{p}\leqslant s$, we derive from the embedding $H^s(\mathbb{R}^d)\hookrightarrow L^p(\mathbb{R}^d)$ that

$$\|f\|_{H^{-s}}=\sup{}_{g\in\mathcal{S},\|g\|_{H^s}\leqslant 1}\Big|\int_{\mathbb{R}^d}f\bar g\,\mathrm{d}x\Big|\leqslant\|f\|_{L^{p'}}\sup{}_{g\in\mathcal{S},\|g\|_{H^s}\leqslant 1}\|g\|_{L^p}\leqslant C\|f\|_{L^{p'}}.$$

□

Remark 2.8. *Let $s=1$, then*

$$\begin{aligned}&H^1(\mathbb{R})\hookrightarrow C_0(\mathbb{R}),\quad H^1(\mathbb{R}^2)\hookrightarrow L^p(\mathbb{R}^2),\ \forall p\in[2,\infty),\\ &H^1(\mathbb{R}^3)\hookrightarrow L^p(\mathbb{R}^3),\quad L^{p'}(\mathbb{R}^3)\hookrightarrow H^{-1}(\mathbb{R}^3),\ \forall p\in[2,6].\end{aligned}$$

*(**Exercise**.) Find a function in $H^1(\mathbb{R}^2)$ but not in $L^\infty(\mathbb{R}^2)$.*

Corollary 2.1 (Gagliardo-Nirenberg's inequality)**.** *For any $p\in[2,2^*)$ with $2^*=\begin{cases}\infty & \text{if } d=1,2\\ \frac{2d}{d-2} & \text{if } d\geqslant 3\end{cases}$, there exists a constant C depending on p,d such that the following interpolation inequality holds true*

$$\|f\|_{L^p(\mathbb{R}^d)}\leqslant C\|f\|_{L^2(\mathbb{R}^d)}^{1-\theta}\|\nabla f\|_{L^2(\mathbb{R}^d)}^{\theta},\quad \forall f\in H^1(\mathbb{R}^d),\quad \theta=\frac{d}{2}-\frac{d}{p}.$$

Proof. It follows from the Sobolev embedding $\|f\|_{L^p(\mathbb{R}^d)} \leqslant C\|f\|_{\dot{H}^\theta(\mathbb{R}^d)}$ with $\theta = \frac{d}{2} - \frac{d}{p} \in [0,1)$, $p \in [2, 2^*)$ and the Sobolev interpolation

$$\|f\|_{\dot{H}^\theta}^2 = \int_{\mathbb{R}^d} |\hat{f}|^{2(1-\theta)}(|\xi|^2|\hat{f}|^2)^\theta \,\mathrm{d}\xi \leqslant \|f\|_{L^2}^{1-\theta}\|f\|_{\dot{H}^1}^\theta.$$

□

Theorem 2.8. *Let $s > 0$ and let $p_c = d(\frac{d}{2} - s)^{-1}$ if $s < \frac{d}{2}$ and $p_c = \infty$ if $s \geqslant \frac{d}{2}$. Then the embedding $H^s(\mathbb{R}^d) \hookrightarrow L^p_{\text{loc}}(\mathbb{R}^d)$, $1 \leqslant p < p_c$ is compact in the following sense: For any bounded sequence $(f_n)_n$ in $H^s(\mathbb{R}^d)$, there exists a subsequence $(f_{\psi(n)})_n$ and $f \in H^s(\mathbb{R}^d)$ such that for any compact set $K \subset\subset \mathbb{R}^d$*

$$f_{\psi(n)} \to f \text{ in } L^p(K).$$

Proof. **Step 1** Take the smooth mollifier function: $\varphi \in C_0^\infty(\mathbb{R}^d)$, $\varphi \geqslant 0$, $\int_{\mathbb{R}^d} \varphi \,\mathrm{d}x = 1$ and its rescaled functions $\varphi_\varepsilon(x) = \varepsilon^{-d}\varphi(\varepsilon^{-1}x)$. Then

$$\sup_{\|g\|_{H^s} \leqslant 1} \|\varphi_\varepsilon * g - g\|_{L^2(\mathbb{R}^d)} \to 0 \text{ as } \varepsilon \to 0, \text{ if } s > 0,$$
$$\sup_{\|g\|_{H^s} \leqslant 1} \|\varphi_\varepsilon * g - g\|_{L^\infty(\mathbb{R}^d)} \to 0 \text{ as } \varepsilon \to 0, \text{ if } s > \frac{d}{2}.$$

Step 2 For any fixed $\varepsilon_0 > 0$, for any fixed $R > 0$, the map

$$\varphi_{\varepsilon_0} * : L^2(\mathbb{R}^d) \mapsto L^\infty(\bar{B}_R), \quad \bar{B}_R = \{x \in \mathbb{R}^d \,|\, |x| \leqslant R\}$$

is compact (by Young's inequality and Arzela-Ascoli's theorem).
Step 3 The identity map

$$\text{Id} : H^s(\mathbb{R}^d)(\subset L^2(\mathbb{R}^d)) \mapsto L^2(\bar{B}_R)(\subset L^\infty(\bar{B}_R)), \quad s > 0$$

as the uniform limit of $\varphi_\varepsilon *$ is compact.
Step 4 Since $H^s(\mathbb{R}^d) \hookrightarrow L^{p_c}(\mathbb{R}^d)$ if $s < \frac{d}{2}$, then by interpolation (or Hölder's inequality) $H^s(\mathbb{R}^d) \hookrightarrow\hookrightarrow L^p(\bar{B}_R)$ compactly for all $p \in [1, p_c)$ and Cantor's diagonal argument ensures the compact embedding $H^s(\mathbb{R}^d) \hookrightarrow\hookrightarrow L^p_{\text{loc}}(\mathbb{R}^d)$. Similar result holds for $s \geqslant \frac{d}{2}$. □

Remark 2.9. *Notice that for any $s > 0$,* ***(Exercise)***

$$\sup_{\|g\|_{H^s} \leqslant 1} \|\varphi_\varepsilon * g - g\|_{L^2(\mathbb{R}^d)} \to 0 \text{ as } \varepsilon \to 0,$$
$$\sup_{\|g\|_{L^2} = 1} \|\varphi_\varepsilon * g - g\|_{L^2(\mathbb{R}^d)} \geqslant 1.$$

The compact embedding $H^s(\mathbb{R}^d) \hookrightarrow\hookrightarrow L^p_{\text{loc}}(\mathbb{R}^d)$ with $s \in [0, \frac{d}{2})$, $p \in [1, p_c)$ is optimal in the sense that the embeddings $H^s(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d)$, $2 \leqslant p < p_c$ and $H^s(\mathbb{R}^d) \hookrightarrow L^{p_c}_{\text{loc}}(\mathbb{R}^d)$ are not compact ***(Exercise).*** *We are going to give the concentration-compactness lemma describing the embedding $H^1 \hookrightarrow L^2$ later in the lecture.*

2.3.2 Sobolev spaces $W^{k,p}(\mathbb{R}^d)$

We recall here the Sobolev embedding results for the Sobolev spaces $W^{k,p}(\mathbb{R}^d)$ without proof for the sake of the completeness. Please check the materials in the ,,Functional Analysis" lecture.
Recall the definition of the Sobolev space $W^{k,p}(\mathbb{R}^d)$, $k \geqslant 0$ integers as follows

$$W^{k,p}(\mathbb{R}^d) = \{f \in L^p(\mathbb{R}^d) | \partial^\alpha f \in L^p(\mathbb{R}^d),\, 0 \leqslant |\alpha| \leqslant k\}. \tag{2.50}$$

The Sobolev space $W^{k,p}(\mathbb{R}^d)$, $k \in \mathbb{N}$, $1 \leqslant p \leqslant \infty$ is a Banach space equipped with the norm

$$\|f\|_{W^{k,p}(\mathbb{R}^d)} = \begin{cases} \left(\sum_{|\alpha|\leqslant k} \|\partial^\alpha f\|^p_{L^p(\mathbb{R}^d)}\right)^{\frac{1}{p}} & \text{if } 1 \leqslant p < \infty, \\ \max_{|\alpha|\leqslant k} \|\partial^\alpha f\|_{L^\infty(\mathbb{R}^d)} & \text{if } p = \infty. \end{cases}$$

For $1 \leqslant p < \infty$, the test function space $\mathcal{D}(\mathbb{R}^d)$ is dense in $W^{k,p}(\mathbb{R}^d)$. If $p = 2$, then $W^{k,2}(\mathbb{R}^d) = H^k(\mathbb{R}^d)$ as defined in (2.47) and obviously $W^{k_1,p}(\mathbb{R}^d) \subset W^{k_0,p}(\mathbb{R}^d)$ if $k_0 \leqslant k_1$. We can also define the general Sobolev space $W^{s,p}(\Omega)$, $W_0^{s,p}(\Omega)$, $s \in \mathbb{R}$, $\Omega \subset \mathbb{R}^d$ some open set, which we don't discuss in this lecture. We just keep in mind that in bounded domains Ω, one has always to pay attention to the boundary.
Recall the definition of the Hölder spaces $C^{m,\sigma}(\Omega)$, $m \in \mathbb{N}$, $\sigma \in (0,1)$, $\Omega \subset \mathbb{R}^d$ some open set, as follows

$$\begin{aligned} C^{m,\sigma}(\Omega) &= \{f \in C^m(\Omega) \,|\, \partial^\alpha f \in C^\sigma(\Omega),\, \forall |\alpha| = m\}, \qquad (2.51) \\ C^{0,\sigma}(\Omega) := C^\sigma(\Omega) &= \{f \in C(\Omega) \,|\, \|f\|_{C^\sigma(K)} < \infty,\, \forall K \subset \Omega \text{ compact } \}, \end{aligned}$$

where

$$\|f\|_{C^\sigma(\overline{K})} = \|f\|_{L^\infty(K)} + \sup_{x,y\in K,\, x\neq y} \frac{|f(x) - f(y)|}{|x-y|^\sigma}.$$

Similarly we can define

$$\begin{aligned} C^{m,\sigma}(\overline{\Omega}) &= \{f \in C^m(\overline{\Omega}) \,|\, \partial^\alpha f \in C^\sigma(\overline{\Omega}),\, \forall |\alpha| = m\}, \\ C^{0,\sigma}(\overline{\Omega}) := C^\sigma(\overline{\Omega}) &= \{f \in C(\overline{\Omega}) \,|\, \|f\|_{C^\sigma(\bar{\Omega})} < \infty\}. \end{aligned}$$

We also have the following Sobolev embedding theorem for $W^{k,p}(\mathbb{R}^d)$ which we don't prove in this lecture:

Theorem 2.9. *Let $k \in \mathbb{N}^*$, $1 \leqslant p < \infty$. Then the following Sobolev embedding results hold true:*

- *If* $1 \leqslant p < \frac{d}{k}$, *then* $W^{k,p}(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d)$ *for any* $q \in [p, p_c]$ *with* $\frac{d}{p} - k = \frac{d}{p_c}$ *continuously and there exists a constant* C *depending on* d, k, p *such that*

$$\|f\|_{L^{p_c}(\mathbb{R}^d)} \leqslant C \sum_{|\alpha|=k} \|\partial^\alpha f\|_{L^p(\mathbb{R}^d)};$$

- *If* $p = \frac{d}{k}$, *then* $W^{k,\frac{d}{k}}(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d)$ *for any* $q \in [p, \infty)$ *continuously;*

- *If* $\max(1, \frac{d}{k}) < p < \infty$, *then* $W^{k,p}(\mathbb{R}^d) \hookrightarrow C^{m,\sigma}(\mathbb{R}^d) \cap C_0(\mathbb{R}^d)$, $m = [k - \frac{d}{p}]$, $\sigma = k - m$.

Furthermore, the embeddings are compact in the local sense as in Theorem 2.8: For example, let $k = 1$, $p^* = \begin{cases} \frac{dp}{d-p} & \text{if } p < d \\ \infty & \text{otherwise} \end{cases}$, *then the embedding* $W^{1,p}(\mathbb{R}^d) \hookrightarrow\hookrightarrow L^q_{\text{loc}}(\mathbb{R}^d)$ *is compact for* $q \in [1, p^*)$.

[30.05.2022]

[03.06.2022]

2.4 H^1 theory

2.4.1 Local well-posedness in H^1

Theorem 2.10 (LWP in H^1)**.** *Let* $1 < p < 2^* - 1$ *(i.e.* $1 < p < \infty$ *if* $d = 1, 2$ *and* $1 < p < 1 + \frac{4}{d-2}$ *if* $d \geqslant 3$*) be a* H^1 *subcritical exponent. Let* $\kappa = \pm 1$. *Let* $u_0 \in H^1(\mathbb{R}^d)$.
Then (NLS) *is locally well-posed in* $H^1(\mathbb{R}^d)$*: There exists a positive time* $T > 0$ *depending on* $\|u_0\|_{H^1}, p, d$, *a unique solution*

$$u \in Y_T = \{u \in C([-T, T]; H^1(\mathbb{R}^d)) \,|\, u \in L^q([-T, T]; W^{1,\rho}(\mathbb{R}^d))\}$$

for every admissible exponent pair (q, ρ), *and there exists a neighborhood* V *of* u_0 *in* H^1 *such that the flow map*

$$\Phi : V \mapsto Y_T \text{ via } u_0 \mapsto u$$

is Lipschitzian continuous.

Proof. We are going to show that the nonlinear map $\Psi : u \mapsto \Psi(u)$ given by (2.40) is a well-defined contraction map in the complete metric space

$$Y_T(R) := \{u \in Y_T | \|u\|_T := \|u\|_{L^\infty_T H^1} + \|u\|_{L^q_T L^\rho} + \|\nabla u\|_{L^q_T L^\rho} \leqslant R\}$$

with appropriately chosen admissible exponent pair (q, ρ) and T, R (depending on $\|u_0\|_{H^1}, p, d)$. Here we denote $\|u\|_{L^q_T Y} := \big\|\|u(t,\cdot)\|_Y\big\|_{L^q([-T,T])}$.
For $d \geqslant 3$, by Strichartz estimates,

$$\|\Psi(u)\|_{L^\infty_T L^2 \cap L^q_T L^\rho} \leqslant C\|u_0\|_{L^2} + C\||u|^{p-1}u\|_{L^{q'}_T L^{\rho'}}.$$

Similarly,

$$\begin{aligned}\|\nabla\Psi(u)\|_{L^\infty_T L^2 \cap L^q_T L^\rho} &\leqslant C\|\nabla u_0\|_{L^2} + C\|\nabla(|u|^{p-1}u)\|_{L^{q'}_T L^{\rho'}} \\ &\leqslant C\|\nabla u_0\|_{L^2} + C\Big\|\|u\|^{p-1}_{L^r}\|\nabla u\|_{L^\rho}\Big\|_{L^{q'}_T}, \quad \frac{p-1}{r} + \frac{1}{\rho} = \frac{1}{\rho'}.\end{aligned}$$

The idea would be to use Sobolev embedding $\|u\|_{L^r(\mathbb{R}^d)} \leqslant C\|u\|_{W^{1,\rho}(\mathbb{R}^d)}$, where $\frac{1}{r} = \frac{1}{\rho} - \frac{1}{d} \in (0, \frac{1}{\rho})$ if $\rho \in (0, d)$ holds, such that

$$\|\Psi(u)\|_T = \|\Psi(u)\|_{L^\infty_T L^2 \cap L^q_T L^\rho} + \|\nabla\Psi(u)\|_{L^\infty_T L^2 \cap L^q_T L^\rho} \leqslant C(\|u_0\|_{H^1} + T^{\frac{1}{q'} - \frac{p}{q}}\|u\|^p_T).$$

If $d \geqslant 3$, the above estimate holds true if the indices q, ρ, r satisfy

$$\frac{2}{q} + \frac{d}{\rho} = \frac{d}{2}, \quad \rho, q \geqslant 2, \quad \frac{p-1}{r} + \frac{1}{\rho} = \frac{1}{\rho'}, \quad \frac{1}{r} = \frac{1}{\rho} - \frac{1}{d}, \quad \rho \in [2, d),$$

that is, if $p + 1 < 2^*$,

$$\frac{1}{\rho} = \frac{1}{d} + \frac{1}{p+1}(1 - \frac{2}{d}) > \frac{1}{d}, \quad \frac{1}{q} = \frac{d-2}{2}(\frac{1}{2} - \frac{1}{p+1}) \in (0, \frac{1}{2^*}).$$

Therefore for the above chosen admissible pair (q, ρ) with $\theta = 1 - \frac{p+1}{q} \in (0, 1)$, we have

$$\|\Psi(u)\|_T \leqslant C\|u_0\|_{H^1} + CT^\theta\|u\|^p_T,$$

and we can choose

$$R = C_1\|u_0\|_{H^1}, \quad T = C_1^{-1}\|u_0\|_{H^1}^{-\frac{p-1}{\theta}}$$

for some large enough constant C_1 such that Ψ is a contractive mapping in $Y_T(R)$. Since $|u|^{p-1}u \in L^{q'}_t([-T, T]; W^{1,\rho'}_x)$ with (q, ρ) the above admissible exponent pair, the unique fixed point indeed belongs to $L^{q_1}([-T, T]; W^{1,\rho_1})$ for any admissible exponent pair (q_1, ρ_1) by Strichartz estimates.
For $d = 1, 2$, for any $1 < p < \infty$, similarly as above we can show that the map Ψ is contractive in

$$\{u \in C([-T, T]; H^1) \cap L^{q_0}([-T, T]; W^{1,\rho_0}) | \|u\|_T = \|u\|_{L^\infty_T H^1} + \|u\|_{L^{q_0}_T W^{1,\rho_0}} \leqslant R\}$$

for appropriately chosen admissible exponent pair (q_0, ρ_0) with $q_0 > p \geqslant \rho_0/2 > 1$ and R, T. (**Exercise**.) □

2.4.2 Global well-posedness in H^1

Theorem 2.11 (GWP in H^1)**.** *Assume the hypotheses in Theorem 2.10. Then the solution obtained in Theorem 2.10 can be extended uniquely globally in time if*

- *in the defocusing case* $\kappa = 1$*;*
- *in the focusing case* $\kappa = -1$ *and* $1 < p < 1 + \frac{4}{d}$*;*
- *in the focusing case* $\kappa = -1$*,* $p = 1 + \frac{4}{d}$ *and* $\|u_0\|_{L^2} < c_0$ *with* c_0 *some fixed constant;*
- *in the focusing case* $\kappa = -1$*,* $1 + \frac{4}{d} < p$ *and* $\|u_0\|_{H^1} \leqslant \varepsilon_0$ *with* ε_0 *some sufficiently small constant,*

such that

$$u \in C(\mathbb{R}; H^1_x) \cap L^q_{\text{loc}}(\mathbb{R}; W^{1,\rho}(\mathbb{R}^d)) \text{ with admissible exponent pair } (q, \rho), \tag{2.52}$$

$$M(u(t)) = M(u_0), \quad E(u(t)) = E(u_0), \forall t \in \mathbb{R},$$

where

$$M(u) = \int_{\mathbb{R}^d} |u|^2 \,\mathrm{d}x\,, \quad E(u) = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla u|^2 \,\mathrm{d}x + \frac{\kappa}{p+1}\int_{\mathbb{R}^d} |u|^{p+1} \,\mathrm{d}x\,.$$

Proof. **Step 1** Mass and energy conservation laws on $[-T, T]$
Recall the proof of Theorem 2.4:

$$\operatorname{Im}\int_0^t \int_{\mathbb{R}^d} \bar{u}_{m,n} \cdot (\text{NLS})_{m,n} \Rightarrow M(u(t)) = M(u_0), \quad \forall t \in [-T, T],$$

where we made use of the following facts for some appropriate $r \in [2, \infty)$:

$$u \in C([-T, T]; L^2_x) \cap L^q([-T, T]; L^r_x), \quad |u|^{p-1}u \in L^{q'}([-T, T]; L^{r'}_x).$$

Recall in the proof of the H^1-LWP result in Theorem 2.10 that the solution satisfies

$$u \in C([-T, T]; H^1) \cap L^q_T W^{1,\rho}, \quad |u|^{p-1}u \in L^{q'}_T W^{1,\rho'},$$

which implies the mass conservation law immediately.
We follow the same procedure to show the conservation of the energy:

$$\operatorname{Im}\int_0^t \int_{\mathbb{R}^d} (-\Delta\bar{u}_{m,n} + \kappa|u_{m,n}|^{p-1}\bar{u}_{m,n}) \cdot (\text{NLS})_{m,n} \Rightarrow E(u(t)) = E(u_0), \forall t \in [-T, T].$$

Indeed, for $d \geqslant 3$, we have from the proof of Theorem 2.10, the Sobolev embedding results in Theorem 2.9 and the interpolation results in Lebesgue spaces (i.e. log-convexity of L^p norms) $\|f\|_{L^{p_\theta}} \leqslant \|f\|_{L^{p_0}}^{1-\theta}\|f\|_{L^{p_1}}^{\theta}$ if $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ that

$$
\begin{aligned}
&u \in \left(L_T^\infty H^1 \cap L_T^q W^{1,\rho}\right) \subset \left(L_T^\infty L^{(\frac{1}{2}-\frac{1}{d})^{-1}} \cap L_T^q L^{(\frac{1}{\rho}-\frac{1}{d})^{-1}}\right) \subset L_T^{p\alpha} L^{p(\frac{1}{\rho}+\frac{1}{d})^{-1}},\\
&|u|^{p-1}u \in L_T^{q'} W^{1,\rho'} \subset L_T^{q'} L^{(\frac{1}{\rho'}-\frac{1}{d})^{-1}},
\end{aligned}
$$

for some

$$
\alpha = \frac{q}{p} \frac{\frac{1}{2}-\frac{1}{\rho}}{\left(\frac{1}{2}-\frac{1}{d}\right) - \frac{1}{p}\left(\frac{1}{\rho}+\frac{1}{d}\right)} > q \text{ since } 1 < p < 1 + \frac{4}{d-2}.
$$

Then we can assume

$$
\begin{aligned}
&u_{m,n} \to u \text{ in } L_T^q W^{1,\rho},\\
&|u_{m,n}|^{p-1}u_{m,n},\ (|u|^{p-1}u)_{m,n} \to |u|^{p-1}u \text{ in } L_T^{q'} W^{1,\rho'} \cap L_T^{q'} L^{(\frac{1}{\rho'}-\frac{1}{d})^{-1}} \cap L_T^{q} L^{(\frac{1}{\rho}+\frac{1}{d})^{-1}},
\end{aligned}
$$

which implies that if we take the limit in

$$
\operatorname{Im} \int_0^t \int_{\mathbb{R}^d} (-\Delta \bar{u}_{m,n} + \kappa |u_{m,n}|^{p-1}\bar{u}_{m,n}) \cdot (\mathrm{NLS})_{m,n},
$$

then $E(u(t)) = E(u_0)$, $\forall t \in [-T, T]$. Similarly we have the energy conservation law for $d = 1, 2$.

[03.06.2022]

[13.06.2022]

Step 2 If $\kappa = 1$, then recalling the mass and energy conservation laws we have the uniform bound

$$
\|u(t)\|_{H_x^1}^2 \leqslant M(u_0) + 2E(u_0)
$$

on the time interval $[-T, T]$.
Since the existence time T only depends on $p, d, \|u_0\|_{H^1}$ and more precisely $T = C^{-1}\|u_0\|_{H^1}^{-\frac{p-1}{\theta}}$ for some big enough constant C, the solution obtained in Theorem 2.10 can be extended uniquely to all the times.
Step 3 If $\kappa = -1$, then by the Gagliardo-Nirenberg's inequality in Corollary 2.1 for $p + 1 < 2^*$, i.e. $p < 1 + \frac{4}{d-2}$ the H^1 subcritical exponent,

$$
\|u\|_{L^{p+1}(\mathbb{R}^d)} \leqslant C_0 \|u\|_{L^2(\mathbb{R}^d)}^{1-\gamma} \|\nabla u\|_{L^2(\mathbb{R}^d)}^{\gamma}, \quad \gamma = \frac{d}{2} - \frac{d}{p+1} \in (0, 1), \tag{2.53}
$$

we obtain from the energy conservation law that

$$\begin{aligned}\frac{1}{2}\|\nabla u(t)\|^2_{L^2_x(\mathbb{R}^d)} &\leqslant E(u_0) + \frac{1}{p+1}\|u(t)\|^{p+1}_{L^{p+1}_x(\mathbb{R}^d)}\\ &\leqslant E(u_0) + C\|u(t)\|^{(p+1)(1-\gamma)}_{L^2_x(\mathbb{R}^d)}\|\nabla u(t)\|^{(p+1)\gamma}_{L^2_x(\mathbb{R}^d)}\\ &\leqslant E(u_0) + C\|u(t)\|^{(p+1)(1-\gamma)(1-\frac{(p+1)\gamma}{2})^{-1}}_{L^2_x(\mathbb{R}^d)} + \frac{1}{4}\|\nabla u(t)\|^2_{L^2_x(\mathbb{R}^d)},\end{aligned}$$

if

$$(p+1)\gamma = \frac{d}{2}(p+1) - d < 2 \text{ i.e. } 1 < p < 1 + \frac{4}{d}.$$

By the mass conservation law, we obtain the uniform bound on $\|u(t)\|_{H^1_x}$ on the existence time interval and hence the global well-posedness holds true in the mass subcritical case.

Step 4 If $\kappa = -1$ and $p = 1 + \frac{4}{d}$, then the above inequality is replaced by

$$\begin{aligned}\frac{1}{2}\|\nabla u(t)\|^2_{L^2_x(\mathbb{R}^d)} &\leqslant E(u_0) + \frac{1}{2+\frac{4}{d}}\|u(t)\|^{p+1}_{L^{p+1}_x(\mathbb{R}^d)}\\ &\leqslant E(u_0) + \frac{1}{2+\frac{4}{d}}C_0^{2+\frac{4}{d}}\|u(t)\|^{\frac{4}{d}}_{L^2_x(\mathbb{R}^d)}\|\nabla u(t)\|^2_{L^2_x(\mathbb{R}^d)}.\end{aligned}$$

Thus if $\|u_0\|_{L^2_x} < c_0$ some fixed constant (such that $\frac{1}{2+\frac{4}{d}}C_0^{2+\frac{4}{d}}c_0^{\frac{4}{d}} = \frac{1}{2}$) then the solution still extends globally in time.

Step 5 If $\kappa = -1$ and $p \in (1 + \frac{4}{d}, 2^* - 1)$ is mass supercritical and energy subcritical, then $(p+1)\gamma > 2$ and we can assume the smallness condition $\|u_0\|_{H^1_x} \leqslant \varepsilon_0$ such that $\|u(t)\|_{H^1_x} \leqslant 2\varepsilon_0$ globally in time for sufficiently small ε_0 (by bootstrap arguments).

□

Remark 2.10. *It was proved in [Weinstein '1983 CMP] that if* $p = 1 + \frac{4}{d}$*, then*

$$\begin{aligned}&\inf_{f\in H^1}\frac{\|f\|^{\frac{4}{d}}_{L^2(\mathbb{R}^d)}\|\nabla f\|^2_{L^2(\mathbb{R}^d)}}{\|f\|^{2+\frac{4}{d}}_{L^{2+\frac{4}{d}}(\mathbb{R}^d)}} = \frac{\|Q\|^{\frac{4}{d}}_{L^2(\mathbb{R}^d)}}{1+\frac{2}{d}},\\ &\text{i.e. } \frac{1}{2+\frac{4}{d}}\|f\|^{2+\frac{4}{d}}_{L^{2+\frac{4}{d}}(\mathbb{R}^d)} \leqslant \frac{1}{2}\frac{\|f\|^{\frac{4}{d}}_{L^2}}{\|Q\|^{\frac{4}{d}}_{L^2}}\|\nabla f\|^2_{L^2},\end{aligned} \tag{2.54}$$

where Q is the unique positive radial solution of (1.22). *It follows from* (2.54) *that*

$$E(u) = \frac{1}{2}\|\nabla u\|^2_{L^2} - \frac{1}{2+\frac{4}{d}}\|u\|^{2+\frac{4}{d}}_{L^{2+\frac{4}{d}}} \geqslant \frac{1}{2}\Big(1 - \frac{\|u\|^{\frac{4}{d}}_{L^2}}{\|Q\|^{\frac{4}{d}}_{L^2}}\Big)\|\nabla u\|^2_{L^2}, \tag{2.55}$$

and in particular $E(Q) = 0$*. If* $\|u_0\|_{L^2(\mathbb{R}^d)} < \|Q\|_{L^2(\mathbb{R}^d)} = c_0$ *then the focusing* (NLS) *in the mass critical case is globally well-posed.*

Remark 2.11. *We can prove the local-in-time well-posedness results in* $H^2(\mathbb{R}^d)$ *for the case* $\begin{cases} 1 < p < \frac{d}{d-4} \text{ if } d \geqslant 5 \\ 1 < p < \infty \text{ if } d \leqslant 4 \end{cases}$*, such that the solution stays in* $C([-T,T]; H^2(\mathbb{R}^d)) \cap L^q([-T,T]; W^{2,\rho}(\mathbb{R}^d))$ *with* (q,ρ) *admissible exponent pair.*
We can also consider the general Sobolev space $H^s(\mathbb{R}^d)$*,* $0 < s < \min\{1, \frac{d}{2}\}$ *with* $1 < p < 1 + \frac{4}{d-2s}$ *and so on.*
There are global well-posedness and scattering results for the energy-critical defocusing nonlinear Schrödinger equation: $p = 1 + \frac{4}{d-2}$*,* $d \geqslant 3$*,* $\kappa = 1$*. See e.g. Colliander-Keel-Staffilani-Takaoka-Tao Ann. Math. 2008 for the case* $d = 3$*,* $p = 5$*.*

Remark 2.12. *We can also show the existence result by compactness method (instead of Banach fixed point theorem here):*

- *Step 1: Construct a sequence of approximate smooth solutions* u_ε *(by regularising* (NLS)*), which satisfy uniform estimates (e.g.* $\|u_\varepsilon\|_{L^p_T(X)} \leqslant C < \infty$*);*
- *Step 2: Pass to the limit by some compactness argument which comes usually from the uniform bound for the time derivatives* $\partial_t u_\varepsilon$*, e.g. by Aubin-Lions' Lemma, if* $X \hookrightarrow\hookrightarrow Y \hookrightarrow Z$*,* $\|u_\varepsilon\|_{L^p_T(X)} + \|\partial_t u_\varepsilon\|_{L^q_T(Z)} \leqslant C$*, then* $u_\varepsilon \to u$ *in* $L^p_T(Y)$ *if* $p < \infty$ *or* $u_\varepsilon \to u$ *in* $C([0,T];Y)$ *if* $p = \infty$ *and* $q > 1$*, such that the strong limit* u *solves* (NLS)*.*

The above procedure is a quite standard way to show the existence result, nevertheless the uniqueness/continuity results are not ensured a priori and their proofs need other arguments.
Here, we may follow the above procedure to show the well-posedness result for (NLS) *and we have used the idea to show the mass/energy conservation laws. The solutions obtained by contraction argument are usually called strong solutions which are unique, continuously depending on the initial data, while the solutions obtained by the above compactness method are usually called weak solutions which could exist all the times but are possibly not unique. Sometimes the strong solutions and the weak solutions coincide.*

2.4.3 The virial space case

Let us define the virial space

$$\Sigma = \{u \in H^1(\mathbb{R}^d) \mid xu \in L^2(\mathbb{R}^d)\} = H^1(\mathbb{R}^d) \cap L^2(|x|^2 \, \mathrm{d}x\,), \tag{2.56}$$

consisting of H^1-functions which decay faster than L^2-functions at infinity. We also define the associated norm as

$$\|u\|_\Sigma = (\|u\|_{H^1_x}^2 + \|xu\|_{L^2_x}^2)^{\frac{1}{2}}.$$

Define the partial differential operator P as

$$P = P(t) = x + 2it\nabla, \quad P_j = x_j + 2it\partial_{x_j}, \quad j = 1, \cdots, d.$$

The operator P has the following properties:

- $P : \mathcal{S}(\mathbb{R} \times \mathbb{R}^d) \mapsto \mathcal{S}(\mathbb{R} \times \mathbb{R}^d)$ and by duality $P : \mathcal{S}'(\mathbb{R} \times \mathbb{R}^d) \mapsto \mathcal{S}'(\mathbb{R} \times \mathbb{R}^d)$. In particular,

$$P(t)w = 2ite^{i\frac{|x|^2}{4t}}\nabla(e^{-i\frac{|x|^2}{4t}}w), \quad w \in \mathcal{S}'(\mathbb{R} \times \mathbb{R}^d), \quad t \neq 0. \tag{2.57}$$

- P and $i\partial_t + \Delta$ commutes:

$$\begin{aligned}[P; i\partial_t + \Delta] &= (x + 2it\nabla)(i\partial_t + \Delta) - (i\partial_t + \Delta)(x + 2it\nabla) \\ &= -i\partial_t(2it)\nabla - 2\nabla = 0.\end{aligned}$$

- For any $g \in \Sigma$,

$$P(t)S(t)g = S(t)xg, \tag{2.58}$$

  and $S(t)g \in \Sigma$ for all $t \in \mathbb{R}$.

  Indeed, if $u = S(t)g$, $g \in \Sigma$ solves the free Schrödinger equation $i\partial_t u + \Delta u = 0$, $u|_{t=0} = g$, then $P(t)u$ satisfies also the free Schrödinger equation with the initial data xg which itself has a unique solution $S(t)(xg)$.

  Furthermore, $S(t)g \in L^2(\mathbb{R}^d)$, $\nabla S(t)g = S(t)\nabla g \in L^2(\mathbb{R}^d)$, and $xS(t)g = -2it\nabla S(t)g + S(t)xg \in L^2(\mathbb{R}^d)$. Hence $S(t) : \Sigma \mapsto \Sigma$.

  More generally, $S(t) : \mathcal{S}(\mathbb{R}^d) \mapsto \mathcal{S}(\mathbb{R}^d)$, $t \in \mathbb{R}$, since

  - $S(t) : H^\infty(\mathbb{R}^d) \mapsto H^\infty(\mathbb{R}^d)$, $H^\infty(\mathbb{R}^d) = \cap_{k \geqslant 0} H^k(\mathbb{R}^d)$;
  - for any $g \in \mathcal{S}(\mathbb{R}^d)$, $x^\alpha g \in H^\infty(\mathbb{R}^d)$ for all multiindices α:

$$\begin{aligned}&xS(t)g = (P - 2it\nabla)S(t)g = S(t)(xg) - 2it\nabla S(t)g \in H^\infty(\mathbb{R}^d), \\ &x_j x_k S(t)g = x_j\big(S(t)(x_k g) - 2it\partial_{x_k} S(t)g\big) \\ &\qquad = S(t)(x_j x_k g) - 2it\partial_{x_j} S(t)(x_k g) - 2itx_j\partial_{x_k} S(t)g \in H^\infty(\mathbb{R}^d), \cdots\end{aligned}$$

By duality $S(t) : \mathcal{S}'(\mathbb{R}^d) \mapsto \mathcal{S}'(\mathbb{R}^d)$.

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- For any $g \in \mathcal{S}'(\mathbb{R}^d)$, (2.58) and the following hold true:

$$S(-t)P(t)g = xS(-t)g. \tag{2.59}$$

Furthermore,

$$\begin{aligned} P(t)S(t-t') &= P(t)S(t)S(-t') = S(t)xS(-t') \\ &= S(t)S(-t')P(t') = S(t-t')P(t') \text{ on } \mathcal{S}'(\mathbb{R}^d). \end{aligned}$$

Theorem 2.12 (Well-posedness in Σ)**.** *Let $p \in (1, 2^* - 1)$ be H^1 subcritial exponent. Let $\kappa = \pm 1$. Let $u_0 \in \Sigma$. Then the Cauchy problem* (NLS) *is locally well-posed in Σ, such that there exists a positive time T depending only on $\|u_0\|_{H^1}, p, d$, a unique solution $u \in C([-T,T];\Sigma) \cap L^q([-T,T];W^{1,\rho})$, $Pu \in L^q([-T,T];L^\rho)$ with admissible exponent pair (q,ρ) and a neibourhood of u_0 in Σ such that the flow map is Lipschitz continuous on it. Furthermore, the global well-posedness result holds true under the four assumptions in Theorem 2.11 respectively.*

Sketchy proof. Recalling the nonlinear mapping Ψ given in (2.40), if

$$u \in Z_T(R, R_1) = \{u \in C([-T,T];\Sigma) \,|\, \|u\|_{L^q_T W^{1,\rho}} \leqslant R, \, \|Pu\|_{L^q_T L^\rho} \leqslant R_1\},$$

for some admissible exponent pair (q,ρ) given in Theorem 2.10, then we derive that

$$\begin{aligned} P(t)\Psi u &= P(t)S(t)u_0 - i\kappa \int_0^t P(t)S(t-t')(|u|^{p-1}u)(t')dt' \\ &= S(t)(xu_0) - i\kappa \int_0^t S(t-t')P(t')(|u|^{p-1}u)(t')dt'. \end{aligned} \tag{2.60}$$

By virtue of (2.57), we derive for $t \neq 0$,

$$|P(|u|^{p-1}u)| = 2|t||\nabla(e^{-i\frac{|x|^2}{4t}}|u|^{p-1}u)| = 2|t|\left|\nabla\left(|e^{-i\frac{|x|^2}{4t}}u|^{p-1}(e^{-i\frac{|x|^2}{4t}}u)\right)\right|,$$

and hence

$$\|P(|u|^{p-1}u)\|_{L^{q'}_T L^{\rho'}} \leqslant \left\|2p|t||u|^{p-1}|\nabla(e^{-i\frac{|x|^2}{4t}}u)|\right\|_{L^{q'}_T L^{\rho'}},$$

which is, by virtue of (2.57) again, bounded by

$$\left\| p|u|^{p-1}|Pu| \right\|_{L_T^{q'} L^{\rho'}}.$$

As in the proof of Theorem 2.10, for $d \geqslant 3$ we have

$$\|P(|u|^{p-1}u)\|_{L_T^{q'} L^{\rho'}} \leqslant p\||u|^{p-1}Pu\|_{L_T^{q'} L^{\rho'}} \leqslant CT^{\frac{1}{q'}-\frac{p}{q}} \|u\|^{p-1}_{L_T^q(W^{1,\rho})} \|Pu\|_{L_T^q L^\rho},$$

and hence we can choose $R = C\|u_0\|_{H_x^1}$, $R_1 = C\|xu_0\|_{L_x^2}$, $T = C^{-1}\|u_0\|_{H_x^1}^{-\theta}$ for C sufficiently large such that Ψ is a contraction mapping in $Z_T(R, R_1)$. □

Remark 2.13. *Noticing* (2.57)*, we can proceed by a recurrence argument to arrive at*

$$P_\alpha = (x + 2itD)^\alpha = (2it)^{|\alpha|} e^{i|x|^2/4t} D^\alpha (e^{-i|x|^2/4t}), \quad [P_\alpha; i\partial_t + \Delta] = 0.$$

Based on the property of the operator P_α, e.g. $d = 1$, $p = 3$,

$$\|P_m(|u|^2 u)\|_{L_x^2} \leqslant C_m \|u\|_{L_x^\infty}^2 \|P_m u\|_{L_x^2}, \quad \|u\|_{L_x^\infty} \leqslant t^{-\frac{1}{2}} \|Pu\|_{L_x^2}^{\frac{1}{2}} \|u\|_{L_x^2}^{\frac{1}{2}},$$

[Hayashi-Nakamitsu-Tsutsumi 1986-1988] proved that if p is an odd integer, $u_0 \in H^m(\mathbb{R}^d) \cap L^2(|x|^k \,\mathrm{d}x)$, $m \geqslant k$, then the regularity and the decay property are both preserved on the existence time interval. In particular, if $u_0 \in \mathcal{S}(\mathbb{R}^d)$, then the solution of (NLS) *$u(t, \cdot) \in \mathcal{S}(\mathbb{R}^d)$ on the existence time interval.*

3 Large time behaviour

In this section we are going to make use of the "variations" of the mass $M(u) = \int_{\mathbb{R}^d} |u|^2 \,\mathrm{d}x$ and the momentum $P_j(u) = \mathrm{Im} \int_{\mathbb{R}} \bar{u}\partial_{x_j} u$ of (NLS):

$$V(u) = \int_{\mathbb{R}^d} |x|^2 |u|^2 \,\mathrm{d}x \,,$$
$$W(u) = \mathrm{Im} \sum_{j=1}^{d} \int_{\mathbb{R}^d} x_j (\bar{u}\partial_{x_j} u) \,\mathrm{d}x \,,$$

to show

- Blowup phenomena, in the focusing mass (super)critical and energy subcritical case: $p \in [1 + \frac{4}{d}, 2^* - 1)$, $\kappa = -1$;
- Scattering phenomena, in the defocusing mass (super)critical and energy subcritical case: $p \in [1 + \frac{4}{d}, 2^* - 1)$, $\kappa = +1$,

provided with regular initial data $u_0 \in \Sigma$.

3.1 Virial and Morawetz identities

We define the virial potential

$$V(u) = \int_{\mathbb{R}^d} |x|^2 |u|^2 \,\mathrm{d}x \,, \tag{3.1}$$

which averages the mass density (with the mass defined in (1.9)) against the weight function $|x|^2$. We define the associated Morawetz action

$$W(u) = \mathrm{Im} \sum_{j=1}^{d} \int_{\mathbb{R}^d} x_j (\bar{u}\partial_{x_j} u) \,\mathrm{d}x \equiv \mathrm{Im} \int_{\mathbb{R}^d} r (\bar{u}\partial_r u) \,\mathrm{d}x \,, \quad r = |x|, \tag{3.2}$$

which averages the momentum densities (with the momentum defined in (1.10)) against the weights (x_j).
Then we have the following Virial and Morawetz identities

Proposition 3.1. *Let $u(t,x)$ be a Schwartz solution of the Cauchy problem* (NLS). *Then*

$$\frac{1}{4}\frac{d}{dt} V(u(t)) = W(u(t)), \tag{3.3}$$

and

$$\frac{1}{2}\frac{d}{dt} W(u(t)) = \int_{\mathbb{R}^d} |\nabla u|^2 \,\mathrm{d}x \, + \kappa \Big(\frac{d}{2} - \frac{d}{p+1}\Big) \int_{\mathbb{R}^d} |u|^{p+1} \,\mathrm{d}x \,. \tag{3.4}$$

Proof. **Exercise.** Making use of the Pohozaev's Identity

$$\begin{aligned}
&\mathrm{Re}\int_{\mathbb{R}^d}\Delta\bar{u}\big(x\cdot\nabla u\big)\,\mathrm{d}x\ = (\frac{d}{2}-1)\int_{\mathbb{R}^d}|\nabla u|^2\,\mathrm{d}x\,,\quad \forall u\in\mathcal{S}(\mathbb{R}^d),\\
&\text{or equivalently, }\mathrm{Re}\int_{\mathbb{R}^d}\Delta\bar{u}\big(\frac{d}{2}u+x\cdot\nabla u\big)\,\mathrm{d}x\ = -\int_{\mathbb{R}^d}|\nabla u|^2\,\mathrm{d}x\,,
\end{aligned}\tag{3.5}$$

where $|\nabla u|^2=\sum_{j=1}^d((\partial_{x_j}\mathrm{Re}\,u)^2+(\partial_{x_j}\mathrm{Im}\,u)^2)$.

□

Remark 3.1. *We can define instead the Virial potential and Morawetz action, with the weights* $|x|^2,(x_j)$ *in* (3.1) *and* (3.2) *replaced by the new weights* $|x|,(\frac{x_j}{|x|})$ *respectively:*

$$\mathcal{V}(u)=\int_{\mathbb{R}^d}|x||u|^2\,\mathrm{d}x\,,$$

$$\mathcal{W}(u)=\mathrm{Im}\sum_{j=1}^d\int_{\mathbb{R}^d}\frac{x_j}{|x|}(\bar{u}\partial_{x_j}u)\,\mathrm{d}x\,.$$

Then we have the following identities (Lin-Strauss' Morawetz Identities) for the Schwartz solution u *of the Cauchy problem* (NLS)*:*

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\mathcal{V}(u(t))=\mathcal{W}(u(t)),\\
&\frac{d}{dt}\mathcal{W}(u(t))=\int_{\mathbb{R}^d}\frac{|\nabla\!\!\!/ u|^2}{|x|}\,\mathrm{d}x\ +\kappa\frac{2(d-1)(p-1)}{p+1}\int_{\mathbb{R}^d}\frac{|u|^{p+1}}{|x|}\,\mathrm{d}x\ -\frac{1}{4}\int_{\mathbb{R}^d}(\Delta^2|x|)|u|^2\,\mathrm{d}x\,,
\end{aligned}$$

where $\nabla\!\!\!/ :=\nabla-\frac{x}{|x|}(\frac{x}{|x|}\cdot\nabla)$ *denotes the angular gradient.*

Corollary 3.1. *Let* $p\in(1,2^*-1)$ *be energy subcritical exponent. Let* $u_0\in\Sigma$ *and let* $u\in C([-T,T];H^1)$, $T<\infty$ *be the solution of* (NLS)*. Then* $u\in C([-T,T];\Sigma)\cap L^q([-T,T];W^{1,\rho})$, $Pu\in L^q([-T,T];L^\rho)$ *for any admissible exponent pair* (q,ρ)*, and the mass and energy conservation laws as well as the virial and Morawetz identities* (3.3)-(3.4) *hold for* u *on the existence time interval* $[-T,T]$*: For any* $t\in[-T,T]$,

$$\begin{aligned}
&M(u(t))=M(u_0),\quad E(u(t))=E(u_0),\\
&\frac{1}{4}V(u(t))-\frac{1}{4}V(u_0)=\int_0^t W(u(t'))dt',\\
&\frac{1}{2}W(u(t))-\frac{1}{2}W(u_0)=\int_0^t\int_{\mathbb{R}^d}|\nabla u|^2\,\mathrm{d}x\,dt+\kappa(\frac{d}{2}-\frac{d}{p+1})\int_0^t\int_{\mathbb{R}^d}|u|^{p+1}\,\mathrm{d}x\,dt.
\end{aligned}$$

Sketchy proof. Recalling the proof of Theorem 2.12, there exists $T_0 > 0$ depending only on $\|u\|_{L^\infty_T(H^1)}$ such that there exists a unique solution $\tilde{u} \in C([t_0-T_0, t_0+T_0]; \Sigma) \cap L^q([t_0-T_0, t_0+T_0]; W^{1,\rho})$, $P\tilde{u} \in L^q([t_0-T_0, t_0+T_0]; L^\rho)$ for any $t_0 \in [-T, T]$, and hence by uniqueness $u = \tilde{u}$ on $[-T, T]$.
We do a regularisation argument and repeat the proof of Proposition 3.1 to arrive at the identities (3.3)-(3.4) for u on $[-T, T]$. (**Exercise**.) □

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3.2 Blowup

Theorem 3.1 (Blowup for the focusing case)**.** *Let $s_c \in [0, 1)$, i.e. $1 + \frac{4}{d} \leqslant p < 2^* - 1$. Let $\kappa = -1$. Let $u_0 \in \Sigma$ with the initial energy $E(u_0) < 0$.*
Then the unique solution $u(t, x)$ obtained in Theorem 2.12 blows up in finite time, and more precisely there exists $T^* < +\infty$ such that

$$\lim_{t \uparrow T^*} \|\nabla u(t)\|_{L^2(\mathbb{R}^d)} = +\infty.$$

Proof. We consider positive time in the following and the negative time can be treated similarly. If the solution $u \in C((a, b); H^1(\mathbb{R}^d))$ on some time interval (a, b), then by the virial and Morawetz identities (3.3)-(3.4), we derive that

$$\begin{aligned} \frac{1}{16}\frac{d^2}{dt^2}V(u) &= \frac{1}{2}\int_{\mathbb{R}^d} |\nabla u|^2 \,\mathrm{d}x - \frac{1}{2}\left(\frac{d}{2} - \frac{d}{p+1}\right)\int_{\mathbb{R}^d} |u|^{p+1} \,\mathrm{d}x \\ &= E(u) - \frac{1}{2}\left(\frac{d}{2} - \frac{d+2}{p+1}\right)\int_{\mathbb{R}^d} |u|^{p+1} \,\mathrm{d}x \leqslant E(u_0) < 0, \end{aligned} \tag{3.6}$$

since $\frac{d}{2} - \frac{d+2}{p+1} = (d+2)(\frac{1}{2+\frac{4}{d}} - \frac{1}{p+1}) \geqslant 0$ if $p \geqslant 1 + \frac{4}{d}$ is mass (super)critical. Hence if $u \in C([0, \infty); H^1(\mathbb{R}^d))$, then the time-dependent quantity $V(u(t)) = \int_{\mathbb{R}^d} |x|^2 |u|^2 \,\mathrm{d}x$ is below a parabola which is negative in finite positive time which is not possible. Thus u blows up at some finite positive time.

By more precise calculations we can derive the upper bounds for the blowup time T^*.

Case $p > 1 + \frac{4}{d}$. We can calculate on the existence time interval

$$\begin{aligned} \frac{1}{16}\frac{d^2}{dt^2}V(u) &= \frac{1}{4}\frac{d}{dt}W(u(t)) \\ &= \left[\frac{1}{2} - \frac{d}{4}\left(\frac{p+1}{2} - 1\right)\right]\int_{\mathbb{R}^d} |\nabla u|^2 \,\mathrm{d}x + \frac{d}{2}\left(\frac{p+1}{2} - 1\right)E(u) \end{aligned}$$

$$< -\alpha \int_{\mathbb{R}^d} |\nabla u|^2 \,\mathrm{d}x\,,$$

where $\alpha = -\frac{1}{2} + \frac{d}{4}(\frac{p+1}{2} - 1) = \frac{d}{8}(p - 1 - \frac{4}{d}) > 0$.
If initially $W(u_0) < 0$, then $W(u(t)) < 0$. Thus $\frac{d}{dt}V(u) < 0$ and $V(u)(t) \leqslant V(u_0)$. Since

$$|W(u)(t)| = -W(u)(t) \leqslant \|ru\|_{L^2(\mathbb{R}^d)} \|\nabla u\|_{L^2(\mathbb{R}^d)} \leqslant (V(u_0))^{\frac{1}{2}} \|\nabla u\|_{L^2(\mathbb{R}^d)},$$

we derive that

$$\frac{1}{4}\frac{d}{dt}(-W(u)) > \alpha (V(u_0))^{-1}(-W(u))^2,$$

and hence

$$(V(u_0))^{\frac{1}{2}} \|\nabla u\|_{L^2(\mathbb{R}^d)} \geqslant -W(u) \geqslant \frac{V(u_0)(-W(u_0))}{V(u_0) + 4\alpha W(u_0) t}$$

from which we derive that $\lim_{t \uparrow T^*} \|\nabla u(t)\|_{L^2(\mathbb{R}^d)} = \infty$ with

$$T^* = (-4\alpha W(u_0))^{-1} V(u_0) = \frac{V(u_0)}{-4\alpha W(u_0)}.$$

If initially $W(u_0) \geqslant 0$, then by virtue of (3.6):

$$\frac{1}{4}\frac{d}{dt}W(u) \leqslant E(u_0) < 0,$$

there exists a positive time t_0 such that $W(u)(t_0) < 0$ and we are in the previous case again.
Case $p = 1 + \frac{4}{d}$. In this critical case, (3.6) gives

$$\frac{1}{16}\frac{d^2}{dt^2}V(u) = \frac{1}{4}\frac{d}{dt}W(u) = E(u_0) < 0.$$

Thus

$$W(u)(t) = W(u_0) + 4E(u_0)t, \quad V(u)(t) = V(u_0) + 4W(u_0)t + 8E(u_0)t^2,$$

and hence there exists a positive time $T^* > 0$ (of order $\frac{W_0 + \sqrt{W_0^2 - 2V_0E_0}}{-4E_0}$) such that $V(u)(T^*) = 0$. By the equality $\|f\|^2_{L^2(\mathbb{R}^d)} = -\frac{1}{d}\sum_{j=1}^d \int_{\mathbb{R}^d} x_j \partial_{x_j}(|f|^2)\,\mathrm{d}x$ for $f \in \mathcal{S}(\mathbb{R}^d)$, we derive the Heisenberg's inequality

$$\|f\|^2_{L^2(\mathbb{R}^d)} \leqslant \frac{2}{d}\sum_{j=1}^d \|x_j f\|_{L^2(\mathbb{R}^d)} \|\partial_{x_j} f\|_{L^2(\mathbb{R}^d)}, \quad \forall f \in \Sigma.$$

Therefore

$$0 < \|u_0\|_{L^2(\mathbb{R}^d)}^2 = \|u(t)\|_{L^2(\mathbb{R}^d)}^2 \leqslant \frac{2}{d}(V(u))^{\frac{1}{2}}\|\nabla u\|_{L^2(\mathbb{R}^d)},$$

which together with $V(u)(T^*) = 0$ implies $\lim_{t\to T^*} \|\nabla u\|_{L^2(\mathbb{R}^d)} = \infty$. □

Corollary 3.2 (Lower bound of the blowup rate)**.** *Let* $d \geqslant 3$, $1 + \frac{4}{d} \leqslant p < 1 + \frac{4}{d-2}$ *and* $u_0 \in H^1(\mathbb{R}^d)$*. Let* $u(t,x)$ *be the solution of the Cauchy problem* (NLS) *satisfying* $\lim_{t\uparrow T^*} \|\nabla u(t)\|_{L^2(\mathbb{R}^d)} = +\infty$*, then*

$$\|\nabla u(t)\|_{L^2(\mathbb{R}^d)} \geqslant C_0(T^* - t)^{-(\frac{1}{p-1}-\frac{d-2}{4})}, \quad \forall t \in [0, T^*).$$

Proof. Recall the proof of Theorem 2.10. For any time $t_0 < T^*$ with $\|u(t_0)\|_{H^1(\mathbb{R}^d)} < \infty$, the solution u with $\|u\|_{L^\infty([t_0,t_0+T];H^1(\mathbb{R}^d))} \leqslant R := C\|u(t_0)\|_{H^1(\mathbb{R}^d)}$ exists at least on the time interval $[t_0, t_0 + T]$, $T > 0$ with

$$T = C^{-1}\|u(t_0)\|_{H^1(\mathbb{R}^d)}^{-\frac{4}{d-2}\frac{p-1}{1+\frac{4}{d-2}-p}} = C^{-1}\|u(t_0)\|_{H^1(\mathbb{R}^d)}^{-\frac{1}{\frac{1}{p-1}-\frac{d-2}{4}}}.$$

Hence

$$T^* - t_0 > T \text{ i.e. } \|u(t_0)\|_{H^1(\mathbb{R}^d)} \geqslant C(T^* - t_0)^{-(\frac{1}{p-1}-\frac{d-2}{4})}.$$

□

Similar results hold for $d = 1, 2$ **Exercise**.

Remark 3.2. *The time* T^* *gives indeed an upper bound for the life span and the solution may blow up before* T^**. We can also make use of the norm* $\|u\|_{L^q}$ *for* $q \geqslant p + 1$ *instead of* $\|\nabla u\|_{L^2}$ *in the estimate of the lifespan. Indeed it is cheap to see that* $\frac{1}{p+1}\|u\|_{L^{p+1}}^{p+1} = \frac{1}{2}\|\nabla u\|_{L^2}^2 - E(u_0) \to \infty$ *as* $t \to T^*$*.*

Remark 3.3 (Blow up rates for the case $p = 1 + \frac{4}{d}$, $\kappa = -1$)**.** *In this case* $\frac{1}{16}\frac{d^2}{dt^2}V(u(t)) = E(u)$*, and by* (2.55)*:* $E(u) \geqslant \frac{1}{2}\Big(1 - \frac{\|u\|_{L^2}^{\frac{4}{d}}}{\|Q\|_{L^2}^{\frac{4}{d}}}\Big)\|\nabla u\|_{L^2}^2$*. If* $E(u_0) < 0$*, then the initial data satisfies* $\|u_0\|_{L^2} > \|Q\|_{L^2}$*.*
We can assume the following assumptions instead of $E(u_0) < 0$ *to derive the blowup results:*

- $E(u_0) = 0$ *and* $W(u_0) < 0$*;*
- $E(u_0) > 0$ *and* $W(u_0) < -\sqrt{E(u_0)V_0(u_0)}$*.*

Pseudo-conformal blow up rate $\|\nabla u\|_{L^2} \underset{t\sim T^*}{\sim} \frac{1}{T^*-t}$

Recall the pseudoconformal invariance in the mass critical case that if $u = u(t,x)$ *is a solution of the nonlinear Schrödinger equation* (NLS)*, then so is* $v(t,x) = \frac{e^{i|x|^2/4t}}{|t|^{\frac{d}{2}}} u(\frac{x}{t}, \frac{1}{t})$*. If* $u_0 \in \Sigma$*, then for any* $t \neq 0$*,* $v(t,\cdot) \in \Sigma$*.*

Let $u(t,x) = e^{it}Q(x)$ *be the solitary solution of the focusing* (NLS)*, then* $v(t,x) = \frac{e^{i(|x|^2+4)/4t}}{|t|^{\frac{d}{2}}} Q(\frac{x}{t})$ *is also a solution in* Σ *for any* $t \neq 0$*, while blows up at* $t = 0$*:* $\|\nabla v(t)\|_{L^2(\mathbb{R}^d)} = O(1/t)$ *as* $t \to 0_+$*.*

Indeed [Merle 1993] showed that the above is the unique minimal mass blow up solution: Let $p = 1 + \frac{4}{d}$*,* $\kappa = -1$ *and* u *be the solution of* (NLS) *with the initial data* $u_0 \in H^1(\mathbb{R}^d)$ *and* $\|u_0\|_{L^2(\mathbb{R}^d)} = \|Q\|_{L^2(\mathbb{R}^d)}$*. If* $\lim_{t\uparrow T} \|\nabla u(t)\|_{L^2} = \infty$*, then up to the symmetries in Subsection 1.1.3*

$$u(t,x) = \frac{e^{i(|x|^2+4)/4(T-t)}}{(T-t)^{\frac{d}{2}}} Q(\frac{x}{T-t}).$$

Let $d = 1,2$*,* $p = 1 + \frac{4}{d}$*,* $\kappa = -1$*,* $w_0 \in H^1(\mathbb{R}^d)$ *such that* $\|w_0\|_{L^2} = \|Q\|_{L^2} + \varepsilon$ *and* $\lim_{t\uparrow T} \|\nabla w(t)\|_{L^2} = \infty$*. [Bourgain-Wang 1997] showed that* $w = u + \varphi$*, where* u *is as above and* φ *remains smooth after the blow up time.*

[Merle 1990 CMP] also proved that for any given $T > 0$*, any set of fixed points* $\{x_1, \cdots, x_k\}$ *in* $\mathbb{R}^d$*, there exists an initial data* u_0 *such that the corresponding solution of the focusing mass critical* (NLS) *blows up exactly at time* T *with the total mass concentrating at the points* $\{x_1, \cdots, x_k\}$ *at the conformal rate.*

[Merle&Raphaël 2018 Ann. ENS] constructed an example of the solution blowing up strictly faster than conformal rate in dimension two: $\|\nabla u\|_{L^2(\mathbb{R}^2)} \sim \frac{|\ln(T^*-t)|}{T^*-t}$*, which concentrates at* K *bubbles at a point* $x_0 \in \mathbb{R}^2$*:* $|u|^2 \rightharpoonup K\|Q\|_{L^2}^2 \delta_{x_0}$*,* $K \geqslant 2$*. This is equivalent (by pseudo-conformal invariance) to the existence of infinite time blowup solutions:* $u(t) \sim e^{i\gamma(t)} \sum_{k=1}^{K} \frac{1}{\lambda(t)} Q(\frac{\cdot - x_k(t)}{\lambda(t)})$*,* $\lambda(t) \sim \frac{1}{\ln t}$*, with* $\|\nabla u(t)\|_{L^2} \sim \ln t$*, as* $t \to \infty$*.*

log-log blow up rate $\|\nabla u\|_{L^2} \underset{t\sim T^*}{\sim} \sqrt{\frac{\ln|\ln(T^*-t)|}{T^*-t}}$

Corollary 3.2 implies that the blow up rate is at least $\|\nabla u(t)\|_{L^2(\mathbb{R}^d)} \geqslant C(T^*-t)^{-\frac{1}{2}}$ *in the mass critical case. Indeed, the numerical simulation suggests the existence of solutions with log-log blow up rate* $\left(\frac{\ln|\ln|T^*-t||}{T^*-t}\right)^{\frac{1}{2}}$*. And when* $d = 1$*, [Perelman 2001] established the existence of a solution with log-log blow up rate.*

[Raphaël 2005] proved that there is a universal gap between the above two blowup rates: Let $\|u_0\|_{L^2} \in (\|Q\|_{L^2}, \|Q\|_{L^2} + \varepsilon)$ *for* $\varepsilon \ll 1$*. Let* u *be the corresponding blowup solution, then either* u *blows up at log-log rate, or* u *blows*

up faster than pseudo-conformal rate, i.e. $\|\nabla u(t)\|_{L^2(\mathbb{R}^d)} \geqslant C(T^* - t)^{-1}$.

[24.06.2022]

[27.06.2022]

3.3 Scattering

Theorem 3.2. *Let* $1+\frac{4}{d} \leqslant p < 2^*-1$ *and* $\kappa = 1$. *Let* $u_0 \in \Sigma$ *and* $u \in C(\mathbb{R};\Sigma)$ *be the global-in-time solution of* (NLS) *given in Theorem 2.12. Then*

$$u \in C(\mathbb{R};\Sigma) \cap L^q(\mathbb{R}; W^{1,\rho}(\mathbb{R}^d)), \text{ with } (q,\rho) \text{ admissible exponent pair,}$$

and u *scatters at large time in the sense that there exist two functions* $u_\pm \in \Sigma$ *such that*

$$\lim_{t\to\pm\infty} \|u(t,\cdot) - S(t)u_\pm\|_\Sigma = 0.$$

Proof. We just show the case $t \to +\infty$ and the case $t \to -\infty$ follows similarly.
Step 1 Pointwise decay
Consider the time-dependent function

$$\begin{aligned} F(t) &= \int_{\mathbb{R}^d} |xu + 2it\nabla u|^2 \,\mathrm{d}x \,+ \frac{8t^2}{p+1}\int_{\mathbb{R}^d} |u|^{p+1}\,\mathrm{d}x \\ &= \int_{\mathbb{R}^d} |x|^2|u|^2\,\mathrm{d}x \,- 4t\mathrm{Im}\int_{\mathbb{R}^d} x\cdot\nabla u\bar{u}\,\mathrm{d}x \,+ 8t^2E(u) \\ &= V(u) - 4tW(u) + 8t^2E(u_0), \end{aligned}$$

where $V(u), W(u), E(u)$ are the Virial potential, Morawetz action and the energy defined in (3.1), (3.2) and (1.11) respectively. By view of the virial and Morawetz identities (3.3)-(3.4), we have

$$\frac{d}{dt}F(t) = \frac{4d\,t}{p+1}[1+\frac{4}{d}-p]\int_{\mathbb{R}^d}|u|^{p+1}\,\mathrm{d}x \,\leqslant 0, \quad \text{ if } p \geqslant 1+\frac{4}{d}.$$

Let $v(t,x) = e^{-i|x|^2/4t}u(t,x)$, then

$$Pu = (x+2it\nabla)u = 2ite^{i\frac{|x|^2}{4t}}\nabla v,$$

and we have

$$\begin{aligned} 8t^2E(v) &= 4t^2\int_{\mathbb{R}^d}|\nabla v|^2\,\mathrm{d}x \,+ \frac{8t^2}{p+1}\int_{\mathbb{R}^d}|v|^{p+1}\,\mathrm{d}x \,= F(t) \\ &\leqslant F(0) = V(u_0). \end{aligned}$$

Hence we have the following pointwise in time decay rate

$$\|\nabla v(t)\|_{L^2(\mathbb{R}^d)} \leqslant (2t)^{-1}(V(u_0))^{\frac{1}{2}}.$$

By Gagliardo-Nirenberg's inequality in Corollary 2.1 and the mass conservation law

$$\|v(t)\|_{L^2_x} = \|u(t)\|_{L^2_x} = \|u_0\|_{L^2_x} = \|v_0\|_{L^2_x},$$

we have the following pointwise decay rate for $\|u\|_{L^r_x(\mathbb{R}^d)}$, $r \in [2, 2^*)$ (in comparison with (2.36) for the linear Schrödinger group $S(t)$) [5]

$$\begin{aligned} \|u(t)\|_{L^r(\mathbb{R}^d)} = \|v(t)\|_{L^r(\mathbb{R}^d)} &\leqslant C\|v(t)\|_{L^2(\mathbb{R}^d)}^{1-(\frac{d}{2}-\frac{d}{r})}\|\nabla v(t)\|_{L^2(\mathbb{R}^d)}^{\frac{d}{2}-\frac{d}{r}} \\ &\leqslant C|t|^{-(\frac{d}{2}-\frac{d}{r})}\|u_0\|_{L^2(\mathbb{R}^d)}^{1-(\frac{d}{2}-\frac{d}{r})}(V(u_0))^{\frac{1}{2}(\frac{d}{2}-\frac{d}{r})}, \quad \forall r \in [2, 2^*). \end{aligned} \tag{3.7}$$

Step 2 Scattering in $L^2(\mathbb{R}^d)$

Recall the Duhamel's formula (Duhamel) for the globally defined solution $u(t, x)$ of (NLS). Then $w(t, \cdot) = S(-t)u(t, \cdot) \in H^1_x(\mathbb{R}^d)$ satisfies

$$w(t) = u_0 - i\int_0^t S(-t'')(|u|^{p-1}u)(t'')dt''.$$

Then for any $0 < t' < t$,

$$w(t) - w(t') = -i\int_{t'}^t S(-t'')(|u|^{p-1}u)(t'')dt'', \tag{3.8}$$

such that by Strichartz estimate (2.39)

$$\|w(t) - w(t')\|_{L^2(\mathbb{R}^d)} \leqslant C\||u|^{p-1}u\|_{L^{q'}([t',t];L^{r'})} = C\Big\|\|u\|^p_{L^{pr'}(\mathbb{R}^d)}\Big\|_{L^{q'}([t',t])}$$

where (q, r) could be any admissible exponent pair. By use of the pointwise decay (3.7) in Step 1, we choose $r = p + 1 < 2^*$, $\frac{2}{q} = \frac{d}{2} - \frac{d}{p+1}$ to arrive at

$$\|w(t) - w(t')\|_{L^2(\mathbb{R}^d)} \leqslant C_0\Big\|(t'')^{-(\frac{d}{2}-\frac{d}{p+1})p}\Big\|_{L^{q'}([t',t])} = C_0\Big(\int_{t'}^t (t'')^{-\frac{2pq'}{q}}dt''\Big)^{\frac{1}{q'}},$$

where C_0 is some constant depending on the initial data $\|u_0\|_\Sigma$. If $p \geqslant 1 + \frac{4}{d}$ then $q \leqslant 2 + \frac{4}{d}$ such that $\frac{2p}{q} > 1$. Hence $\|w(t) - w(t')\|_{L^2(\mathbb{R}^d)} \to 0$ whenever $t', t \to \infty$. Therefore there exists $u_+ \in L^2(\mathbb{R}^d)$ such that $\|u(t) - S(t)u_+\|_{L^2(\mathbb{R}^d)} = \|w(t) - u_+\|_{L^2(\mathbb{R}^d)} \to 0$ as $t \to +\infty$.

[5] We can also derive from the inequality $8t^2E(v) \leqslant V(u_0)$ the decay property $\|v\|_{L^{p+1}} \lesssim t^{-\frac{2}{p+1}}$, which is weaker than the decay estimate (3.7) as $\frac{2}{p+1} \leqslant \frac{d}{2} - \frac{d}{p+1}$ for $p \geqslant 1 + \frac{4}{d}$.

Step 3 Scattering in $H^1(\mathbb{R}^d)$
We first claim that $u \in L^q([0,\infty); W^{1,p+1})$. Indeed, we have already shown $u \in L^q_{\text{loc}}(\mathbb{R}; W^{1,p+1})$ in Theorem 2.11 such that $\|u\|_{L^q([0,T];W^{1,p+1})} \leqslant C(T) < \infty$ for any finite time $T > 0$. For any $t \geqslant T > 0$, by applying Strichartz estimates on the Duhamel's formula (Duhamel) (and also on the spatial derivative of (Duhamel)), we have

$$\begin{aligned}
\|u\|_{L^q([0,t];W^{1,p+1})} &\leqslant C\|u_0\|_{H^1_x} + C\||u|^{p-1}u\|_{L^{q'}([0,T];W^{1,\frac{p+1}{p}})} + C\||u|^{p-1}u\|_{L^{q'}([T;t];W^{1,\frac{p+1}{p}})} \\
&\leqslant C\|u_0\|_{H^1_x} + CT^{\frac{1}{q'}-\frac{p}{q}}\|u\|^p_{L^q([0,T];W^{1,p+1})} + C\|\|u\|^{p-1}_{L^{p+1}_x}\|u\|_{W^{1,p+1}_x}\|_{L^{q'}([T,t])} \\
&\leqslant C\|u_0\|_{H^1_x} + CT^{1-\frac{p+1}{q}}\|u\|^p_{L^q([0,T];W^{1,p+1})} \\
&\quad + C_0\left\|(t'')^{-\frac{2}{q}(p-1)}\right\|_{L^{(\frac{1}{q'}-\frac{1}{q})^{-1}}([T,t])}\|u\|_{L^q([T,t];W^{1,p+1}_x)},
\end{aligned}$$

where we used the pointwise decay estimate (3.7) for the last inequality and we now calculate

$$\begin{aligned}
&\left\|(t'')^{-\frac{2}{q}(p-1)}\right\|_{L^{(\frac{1}{q'}-\frac{1}{q})^{-1}}([T,t])} = C(T^{-\theta} - t^{-\theta}) \leqslant CT^{-\theta} \\
&\text{with } \theta = \frac{2}{q}(p-1) - (\frac{1}{q'} - \frac{1}{q}) = \frac{2}{q}p - 1 > 0 \text{ when } p \geqslant 1 + \frac{4}{d}.
\end{aligned}$$

Hence by choosing T large enough such that $C_0CT^{-\theta} \leqslant \frac{1}{2}$, we have

$$\|u\|_{L^q([0,t];W^{1,p+1})} \leqslant C\|u_0\|_{H^1_x} + CT^{1-\frac{p+1}{q}}\|u\|^p_{L^q([0,T];W^{1,p+1})} \leqslant C(T) < \infty,$$

and as $t \to \infty$ we derive $u \in L^q([0,\infty); W^{1,p+1})$.
We apply spatial derivative and then Strichartz estimate and finally the decay rate (3.7) to (3.8), to arrive at

$$\begin{aligned}
\|\nabla(w(t) - w(t'))\|_{L^2(\mathbb{R}^d)} &\leqslant C\left\|\|u\|^{p-1}_{L^{p+1}(\mathbb{R}^d)}\|\nabla u\|_{L^{p+1}(\mathbb{R}^d)}\right\|_{L^{q'}([t',t])} \\
&\leqslant C_0\left\|(t'')^{-\frac{2}{q}(p-1)}\right\|_{L^{(\frac{1}{q'}-\frac{1}{q})^{-1}}([t',t])}\|\nabla u\|_{L^q([t',t];L^{p+1}(\mathbb{R}^d))}
\end{aligned}$$

which tends to zero whenever $t', t \to \infty$. Therefore $u_+ \in H^1(\mathbb{R}^d)$ such that $\|u(t) - S(t)u_+\|_{H^1(\mathbb{R}^d)} = \|w(t) - u_+\|_{H^1(\mathbb{R}^d)} \to 0$ as $t \to +\infty$.
Step 4 Scattering in Σ
Recall (2.60) when we apply the operator $P = x + 2it\nabla$ to (Duhamel). Then the same argument as in Step 3 implies that (**Exercise**.)

$$\|(x + 2it\nabla)u\|_{L^q([0,\infty);L^{p+1})} < +\infty,$$

and hence by $xS(-t) = S(-t)P(t)$, we arrive at from (3.8) that

$$\begin{aligned}\|(xw)(t) - (xw)(t')\|_{L^2(\mathbb{R}^d)} &= \Big\| \int_{t'}^{t} S(-t')\big(P(|u|^{p-1}u)\big)(t'')dt'' \Big\|_{L^2(\mathbb{R}^d)} \\ &\leqslant C \big\| |u|^{p-1}|Pu| \big\|_{L^{q'}([t',t];L^{(p+1)'})} \to 0 \text{ as } t', t \to \infty.\end{aligned}$$

Therefore $\|xw(t) - xu_+\|_{L^2_x(\mathbb{R}^d)} \to 0$ as $t \to \infty$. □

Remark 3.4 (Scattering results for other cases)**.** *Let us give some further remarks concerning the exponent regime of p in the scattering theory:*

- *We can relax the restriction on $p \in (1, 2^* - 1)$ for the scattering results, nevertheless there are no scattering theory in L^2_x if $p < 1 + \frac{2}{d}$.*

  If $p = 1 + \frac{2}{d}$, $d = 1$, (long range case), and if $S(-t)u(t) - u_- \to 0$ in L^2 as $t \to -\infty$, then $u = u_- = 0$, that is, one can not compare the nonlinear dynamics with the free dynamics. Nevertheless, there are modified scattering results for $p = 3$, $d = 1$, see e.g. [Carles, 2001, CMP].

  For $p \in (1 + \frac{2}{d}, 2^* - 1)$, $\kappa = 1$, there exist scattering states $u_\pm$ in L^2_x. For $p \in (1, 1 + \frac{4}{d})$, then we also have the pointwise decay

  $$\|u(t)\|_{L^r(\mathbb{R}^d)} \leqslant C|t|^{-(\frac{d}{2}-\frac{d}{r})(1-\alpha(r))}, \quad \alpha(r) = \begin{cases} 0 \ \text{if } 2 \leqslant r \leqslant p+1, \\ \frac{(r-p-1)(4-d(p-1))}{(r-2)(4-(d-2)(p-1))} \ \text{if } r > p+1, \end{cases}$$

  by considering the time-dependent quantity $t^2 \int_{\mathbb{R}^d} |v|^{p+1}\,\mathrm{d}x$ and then the quantity $t^2 \int_{\mathbb{R}^d} |\nabla v|^2\,\mathrm{d}x$ via the equality

  $$\frac{d}{dt}(8t^2 E(v)) = \frac{d}{dt}F(u) = \frac{4dt}{p+1}(1 + \frac{4}{d} - p) \int_{\mathbb{R}^d} |v|^{p+1}\,\mathrm{d}x\,.$$

  And if we assume furthermore $p > (2 + d + \sqrt{d^2 + 12d + 4})/(2d)$ such that $2p > q$, then the scattering result in Theorem 3.2 also holds true. Nevertheless if $p \leqslant \frac{2+d+\sqrt{d^2+12d+4}}{2d}$ and u_0 is large or if $p \leqslant 1 + \frac{4}{d+2}$ we do not know whether $u_\pm \in \Sigma$;

- *There is also scattering theory for the focusing case if $p \in (1+\frac{4}{d+2}, 1+\frac{4}{d})$, nevertheless if $p < 1 + \frac{4}{d+2}$ there is no scattering theory in L^2;*

- *We can relax the restriction on the initial data, e.g. $u_0 \in H^1(\mathbb{R}^d)$ such that the scattering theory in the energy space $H^1(\mathbb{R}^d)$ holds for $p \in (1 + \frac{4}{d}, 2^* - 1)$, $d \geqslant 3$, $\kappa = 1$.*

In the L^2-framework, Dodson showed the L^2_x global-in-time wellposedness and scattering result for the defocusing mass-critical case $p = 1 + \frac{4}{d}$, $\kappa = 1$ in [Dodson 2016], and for the focusing mass-critical case $p = 1 + \frac{4}{d}$, $\kappa = -1$ with the mass below the ground state threshold $\|u_0\|_{L^2} < \|Q\|_{L^2}$ in [Dodson 2015, Advances Math.];

Remark 3.5 (Wave operators and scattering operators in scattering theory)**.** *We introduce briefly here the basic notions of scattering theory. Let X be a Banach space. Let $\mathcal{R}_\pm$ be the following two subsets in X:*

$$\begin{aligned}\mathcal{R}_\pm = \{\varphi \in X \mid (\text{NLS}) \textit{ with initial data } \varphi \textit{ has a unique solution} \\ u \textit{ defined for all } t \geqslant 0 (t \leqslant 0) \textit{ such that } u_\pm = \lim_{t\to\pm\infty} S(-t)u(t) \textit{ exists in } X\},\end{aligned}$$

and we call $u_\pm$ the scattering states of φ at $\pm\infty$. Let $U_\pm$ be the following two operators

$$U_\pm : \mathcal{R}_\pm \mapsto X \ \textit{ via } U_\pm(\varphi) = u_\pm = \lim_{t\to\pm\infty} S(-t)u(t).$$

If the mapping $U_\pm$ are injective, we define the wave operators

$$\Omega_\pm = (U_\pm)^{-1} : \mathcal{U}_\pm \mapsto \mathcal{R}_\pm, \quad \mathcal{U}_\pm = U_\pm(\mathcal{R}_\pm) \textit{ via } \Omega_\pm(u_\pm) = \varphi.$$

Let $\mathcal{O}_\pm = U_\pm(\mathcal{R}_+ \cap \mathcal{R}_-)$ and we define the scattering operator $\mathbb{S}$

$$\mathbb{S} = U_+\Omega_- : \mathcal{O}_- \mapsto \mathcal{O}_+ \textit{ via } \mathbb{S}u_- = u_+.$$

Notice that

$$\mathcal{R}_- = \overline{\mathcal{R}_+} := \{\varphi \mid \bar{\varphi} \in \mathcal{R}_+\}, \quad \mathcal{U}_- = \overline{\mathcal{U}_+}, \quad \mathcal{O}_- = \overline{\mathcal{O}_+},$$

and if $\kappa = 0$ the linear Schrödinger equation, then $U_\pm = \Omega_\pm = \mathbb{S} = \mathrm{Id}$.
Let $X = \Sigma$ and

$$1 + \frac{4}{d} \leqslant p < 2^* - 1, \quad \kappa = +1. \tag{3.9}$$

Then by Theorem 3.2,

$$\mathcal{R}_\pm = \Sigma, \quad U_\pm : \Sigma \mapsto \Sigma, \quad u_\pm = U_\pm(u_0) = u_0 - i\int_0^{\pm\infty} S(-t')(|u|^{p-1}u)(t')dt',$$

where $u(t)$ is the solution of (NLS) *with initial data $u_0 \in \Sigma$. Inversely we have the following facts: The wave operators $\Omega_\pm : u_\pm \to u_0$ and scattering operator $\mathbb{S} : u_- \mapsto u_+$ exist and read:*
Assume (3.9)*, then for any $u_+ \in \Sigma$ (resp. $u_- \in \Sigma$), there exists a unique $u_0 \in \Sigma$ such that the Cauchy problem* (NLS) *with the initial data u_0 has a unique solution $u \in C(\mathbb{R};\Sigma)$ with $\|S(-t)u(t) - u_+\|_\Sigma \to 0$ (resp. $\|S(-t)u(t) - u_-\|_\Sigma \to 0$) as $t \to \infty$ (resp. $t \to -\infty$). Hence we can define the scattering operator $\mathbb{S} : \Sigma \mapsto \Sigma$, $\mathbb{S}u_- = U_+\Omega_-u_-$.*

4 Solitary waves

4.1 A minimiser problem

In this subsection we are concerned with the minimisation problem

$$I_M = \inf\Big\{ \|u\|^2_{H^1(\mathbb{R}^d)} \,|\, u \in H^1_r(\mathbb{R}^d), \int_{\mathbb{R}^d} |u|^{p+1} \,\mathrm{d}x = M \Big\}, \tag{4.10}$$

for some $M > 0$, $d \geqslant 2$, and $p \in (1, 2^* - 1)$. We will first give some compact Sobolev embedding results in the setting of radial H^1-function space $H^1_r(\mathbb{R}^d)$, and then show that the minimising sequence converge to the minimiser, which satisfies the corresponding Euler-Lagrangian equation and some regularity and decay properties.

4.1.1 Compact minimisation

Let $d \geqslant 2$. Let $H^1_r(\mathbb{R}^d)$ be the set of the radial functions in $H^1(\mathbb{R}^d)$:

$$H^1_r(\mathbb{R}^d) = \{ f \in H^1(\mathbb{R}^d) \,|\, \exists \tilde{f} : [0,\infty) \to \mathbb{C} \text{ s.t. } f(x) = \tilde{f}(r),\ r = (\sum_{j=1}^d |x_j|^2)^{\frac{1}{2}} \}.$$

It can also be viewed as the completement of the set of the radial functions in $C_0^\infty(\mathbb{R}^d)$ with respect to the H^1-norm:

$$\|f\|^2_{H^1(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |\nabla f|^2 + |f|^2 \,\mathrm{d}x = \omega_d \int_0^\infty (|\partial_r \tilde{f}(r)|^2 + |\tilde{f}(r)|^2) r^{d-1} dr,$$

where ω_d is the area of the unit sphere in $\mathbb{R}^d$.

Lemma 4.1 (Regularity and vanishing property of H^1_r-functions)**.** *Let $d \geqslant 2$ and $u \in H^1_r(\mathbb{R}^d)$. Then $\tilde{u} \in C^{\frac{1}{2}}((0,\infty))$ and*

$$\|r^{\frac{d-1}{2}} u\|_{L^\infty(\mathbb{R}^d)} \leqslant C \|u\|_{H^1(\mathbb{R}^d)}. \tag{4.11}$$

Proof. Let $\varphi(x) = \tilde{\varphi}(r) \in C_0^\infty(\mathbb{R}^d)$. Then

$$\tilde{\varphi}^2(r) = -2 \int_r^\infty \tilde{\varphi}'(\rho) \tilde{\varphi}(\rho) d\rho,$$

and hence

$$|\tilde{\varphi}^2|(r) \leqslant \frac{2}{r^{d-1}} \int_r^\infty |\tilde{\varphi}'\tilde{\varphi}(\rho)|\rho^{d-1} d\rho$$
$$\leqslant \frac{2}{r^{d-1}} \Big(\int_r^\infty |\tilde{\varphi}'|^2 \rho^{d-1} d\rho\Big)^{\frac{1}{2}} \Big(\int_r^\infty |\tilde{\varphi}|^2 \rho^{d-1} d\rho\Big)^{\frac{1}{2}} \leqslant \frac{C}{r^{d-1}} \|\nabla\varphi\|_{L^2(\mathbb{R}^d)}^{\frac{1}{2}} \|\varphi\|_{L^2(\mathbb{R}^d)}^{\frac{1}{2}}.$$

This implies (4.11) by density argument.
Similarly, let $0 < r_1 < r_2 < \infty$ and we calculate by Hölder's inequality

$$|\tilde{\varphi}(r_1) - \tilde{\varphi}(r_2)| \leqslant \Big| \int_{r_1}^{r_2} \tilde{\varphi}' d\rho \Big| \leqslant \frac{1}{r_1^{\frac{d-1}{2}}} \int_{r_1}^{r_2} |\tilde{\varphi}'| \rho^{\frac{d-1}{2}} d\rho$$
$$\leqslant \frac{C}{r_1^{\frac{d-1}{2}}} \|\nabla\varphi\|_{L^2(\mathbb{R}^d)} |r_2 - r_1|^{\frac{1}{2}},$$

which implies that for any compact set $K \subset\subset (0, \infty)$, $\|\tilde{\varphi}\|_{C^{\frac{1}{2}}(\overline{K})} = \|\tilde{\varphi}\|_{L^\infty(K)} + \sup_{r_1 \neq r_2, r_1, r_2 \in K} \frac{|\tilde{\varphi}(r_2) - \tilde{\varphi}(r_1)|}{|r_2 - r_1|^{1/2}} \leqslant C(K)\|\varphi\|_{H^1}$ and hence by density argument $\|\tilde{u}\|_{C^{\frac{1}{2}}(\overline{K})} \leqslant C(K)\|u\|_{H^1}$ which implies $\tilde{u} \in C^{\frac{1}{2}}((0, \infty))$. □

Proposition 4.1 (Compact Sobolev embedding)**.** *Let $d \geqslant 2$ and 2^* as defined in Corollary 2.1. Then the Sobolev embedding $H_r^1(\mathbb{R}^d) \hookrightarrow\hookrightarrow L^q(\mathbb{R}^d)$, $q \in (2, 2^*)$ is compact.*

Proof. Let $u \in H_r^1(\mathbb{R}^d)$ and $2 < q < 2^*$. Then (4.11) implies

$$\int_{|x| \geqslant R} |u|^q \, \mathrm{d}x \leqslant \frac{\|r^{\frac{d-1}{2}} u\|_{L^\infty}^{q-2}}{R^{\frac{(q-2)(d-1)}{2}}} \int_{\mathbb{R}^d} |u|^2 \, \mathrm{d}x \leqslant C R^{-\frac{(q-2)(d-1)}{2}} \|u\|_{H^1(\mathbb{R}^d)}^q$$
$$\to 0 \text{ as } R \to \infty \text{ uniformly for the } H_r^1 \text{ functions with } \|u\|_{H^1} \leqslant 1.$$

This, combined with the compact embedding $H^1(\mathbb{R}^d) \hookrightarrow\hookrightarrow L^q(\bar{B}_R)$ for any $R \in (0, \infty)$ in Theorem 2.8, implies the compact embedding $H_r^1(\mathbb{R}^d) \hookrightarrow\hookrightarrow L^q(\mathbb{R}^d)$. □

Remark 4.1. *The endpoint case $H_r^1(\mathbb{R}^d) \hookrightarrow\hookrightarrow L^2(\mathbb{R}^d)$ does not hold.*

Proposition 4.2 (Compact minimisation)**.** *Let $d \geqslant 2$ and p be a energy-subcritical exponent: $1 < p < 2^* - 1 = \begin{cases} 1 + \frac{4}{d-2} & \text{if } d \geqslant 3 \\ \infty & \text{if } d = 2 \end{cases}$.*
Then for any $M > 0$, the minimisation problem

$$\begin{aligned} &I_M = \inf_{u \in \mathcal{A}_M} \{\|u\|_{H^1(\mathbb{R}^d)}^2\}, \\ &\text{where } \mathcal{A}_M = \Big\{ u \in H_r^1(\mathbb{R}^d) \mid \int_{\mathbb{R}^d} |u|^{p+1} \, \mathrm{d}x = M \Big\}, \end{aligned} \tag{4.12}$$

has a solution $u \in \mathcal{A}_M$ and $I_M > 0$.

Proof. Since by Sobolev's embedding $H^1(\mathbb{R}^d) \hookrightarrow L^{p+1}(\mathbb{R}^d)$, $2 < p+1 < 2^*$: $\|u\|_{H^1(\mathbb{R}^d)} \geqslant C^{-1}\|u\|_{L^{p+1}(\mathbb{R}^d)} = C^{-1}M^{1/(p+1)}$ if $u \in \mathcal{A}_M$, we can take a minimising sequence $(u_n)_n$ in $\mathcal{A}_M$ such that

$$\|u_n\|^2_{H^1(\mathbb{R}^d)} \to I_M \geqslant C^{-2}M^{\frac{2}{p+1}} > 0.$$

Since $(u_n)_n$ are bounded in $H^1_r(\mathbb{R}^d)$, by Proposition 4.1 there exists a subsequence (still denoted by $(u_n)_n$) and $u \in H^1_r(\mathbb{R}^d)$ such that

$$u_n \to u \text{ in } L^{p+1}(\mathbb{R}^d), \quad u_n \rightharpoonup u \text{ in } H^1(\mathbb{R}^d).$$

Therefore

$$\begin{aligned}
&\int_{\mathbb{R}^d} |u|^{p+1}\,\mathrm{d}x = \lim_{n\to\infty}\int_{\mathbb{R}^d} |u_n|^{p+1}\,\mathrm{d}x = M \text{ and thus } u \in \mathcal{A}_M,\\
&\|u\|^2_{H^1(\mathbb{R}^d)} \leqslant \liminf_{n\to\infty}\|u_n\|^2_{H^1(\mathbb{R}^d)} = I_M,
\end{aligned}$$

and hence $u \in \mathcal{A}_M$ is the minimiser of (4.12). □

4.1.2 Euler-Lagrangian equation

Lemma 4.2 (Positivity). *If $u \in \mathcal{A}_M$, then $|u| \in \mathcal{A}_M$, $\||u|\|_{H^1(\mathbb{R}^d)} \leqslant \|u\|_{H^1(\mathbb{R}^d)}$. If $u \in \mathcal{A}_M$ is a minimiser of (4.12), then so is $|u|$ such that $\||u|\|_{H^1(\mathbb{R}^d)} = \|u\|_{H^1(\mathbb{R}^d)}$, and if furthermore $|u| > 0$, then $u = |u|e^{i\gamma}$ for some $\gamma \in \mathbb{R}$.*

Proof. The lemma follows from the following claim that if $u \in H^1(\mathbb{R}^d)$, then

$$\int_{\mathbb{R}^d} |\nabla u|^2\,\mathrm{d}x \geqslant \int_{\mathbb{R}^d} |\nabla |u||^2\,\mathrm{d}x$$

and if $|u| > 0$, then the above equality holds if and only if $u = |u|e^{i\gamma}$ for some $\gamma \in \mathbb{R}$.

Indeed, suppose $u = f + ig$, $f, g \in H^1(\mathbb{R}^d;\mathbb{R})$. Then

$$\begin{aligned}
\int_{\mathbb{R}^d} |\nabla|u||^2\,\mathrm{d}x &= \int_{\mathbb{R}^d}\Big|\frac{f\nabla f + g\nabla g}{\sqrt{f^2+g^2}}\Big|^2\,\mathrm{d}x\\
&= \int_{\mathbb{R}^d}\frac{f^2|\nabla f|^2 + 2fg\nabla f\cdot\nabla g + g^2|\nabla g|^2}{f^2+g^2}\,\mathrm{d}x\\
&= \int_{\mathbb{R}^d}|\nabla f|^2 + |\nabla g|^2\,\mathrm{d}x - \int_{\mathbb{R}^d}\frac{|f\nabla g - g\nabla f|^2}{f^2+g^2}\,\mathrm{d}x\,.
\end{aligned}$$

Hence the above inequality holds and the equality holds if and only if $f\nabla g = g\nabla f$ almost every where. We assume without loss of generality that f, g are continuous. If $g \neq 0$ or equivalently if $g > 0$ is strictly positive or $g < 0$ is strictly negative, then for any $\phi \in C_0^\infty(\mathbb{R}^d)$ we derive

$$\int_{\mathbb{R}^d} \frac{f}{g} \nabla\phi \,\mathrm{d}x = \int_{\mathbb{R}^d} \frac{-g\phi\nabla f + f\phi\nabla g}{g^2} \,\mathrm{d}x = \int_{\mathbb{R}^d} (f\nabla g - g\nabla f)\frac{\phi}{g^2} \,\mathrm{d}x\,,$$

and hence the equality holds if and only if $f = ag$ for some real constant $a \in \mathbb{R}$:

$$u = f + ig = g(a+i) = \sqrt{a^2+1}g\frac{a+i}{\sqrt{a^2+1}},$$

that is, $u = |u|e^{i\gamma}$ for some $\gamma \in \mathbb{R}$, with $|u| = \sqrt{a^2+1}g$, $e^{i\gamma} = \frac{a+i}{\sqrt{a^2+1}}$ if $g > 0$ or $|u| = \sqrt{a^2+1}|g|$, $e^{i\gamma} = -\frac{a+i}{\sqrt{a^2+1}}$ if $g < 0$. If $|u| > 0$ such that $\mathbb{R}^d = f^{-1}(\mathbb{C}\setminus\{0\}) \cup g^{-1}(\mathbb{C}\setminus\{0\})$, f, g continuous, then we can assume $g \neq 0$ without loss of generality. □

Therefore there is no loss of generality that we assume that the minimiser is real and nonnegative.

Proposition 4.3. *Let $u \geqslant 0$ be the minimiser of* (4.12)*. Then there exists $\lambda \in \mathbb{R}$ such that*

$$-\Delta u + u = \lambda u^p. \tag{4.13}$$

The λ in (4.13) *is indeed a positive constant independent on u: $\lambda = I_M/M$.*

[04.07.2022]

[08.07.2022]

Proof. **Step 1 Differentiation**
Let $t \in \mathbb{R}$ and $h \in C_0^\infty(\mathbb{R}^d;\mathbb{R})$ a radial, real-valued function. Then (**Exercise**)

$$\int_{\mathbb{R}^d} |u+th|^{p+1} \,\mathrm{d}x = \int_{\mathbb{R}^d} u^{p+1} \,\mathrm{d}x + (p+1)t\int_{\mathbb{R}^d} hu^p \,\mathrm{d}x + o(|t|) \text{ as } t \to 0.$$

Let h be chosen such that $\int_{\mathbb{R}^d} hu^p \,\mathrm{d}x = 0$, then since $u \in \mathcal{A}_M$, we have

$$\int_{\mathbb{R}^d} |u+th|^{p+1} \,\mathrm{d}x = M + o(|t|).$$

Let $v_t = \frac{M^{\frac{1}{p+1}}}{\|u+th\|_{L^{p+1}}}(u+th)$, then $\|v_t\|_{L^{p+1}} = M^{\frac{1}{p+1}}$, $v_t = (u+th)(1+o(|t|))$ and

$$\|v_t\|_{H^1}^2 = (1+o(t))\Big(\|u\|_{H^1}^2 + 2t\int_{\mathbb{R}^d} (uh + \nabla u \cdot \nabla h) \,\mathrm{d}x + t^2\|h\|_{H^1}^2\Big)$$

$$= \|u\|_{H^1}^2 + 2t \int_{\mathbb{R}^d} (uh + \nabla u \cdot \nabla h) \, \mathrm{d}x \, + o(|t|) \text{ as } t \to 0.$$

Since $u \geqslant 0$ is the minimiser, $\|v_t\|_{H^1} \geqslant \|u\|_{H^1}$ for any $t \in \mathbb{R}$ and hence

$$\int_{\mathbb{R}^d} (uh + \nabla u \cdot \nabla h) \, \mathrm{d}x \, = 0 \text{ for } h \in C_0^\infty(\mathbb{R}^d; \mathbb{R}) \text{ radial,}$$
$$\text{whenever } \int_{\mathbb{R}^d} hu^p \, \mathrm{d}x \, = 0.$$

Step 2 Lagrangian multiplier
Let L_1, L_2 be the two linear forms on the Hilbert space H_r^1 defined by

$$L_1(h) = \int_{\mathbb{R}^d} hu^p \, \mathrm{d}x \, , \quad L_2(h) = \int_{\mathbb{R}^d} (uh + \nabla u \cdot \nabla h) \, \mathrm{d}x \, .$$

Then $\operatorname{Ker} L_1 \subset \operatorname{Ker} L_2$. Let $h \in H_r^1$ with $L_1(h) \neq 0$. Then any $a \in H_r^1$ can be written as

$$a = \frac{L_1(a)}{L_1(h)} h + b \text{ with } b = a - \frac{L_1(a)}{L_1(h)} h \in \operatorname{Ker} L_1 \subset \operatorname{Ker} L_2.$$

Hence $L_2(a) = \frac{L_1(a)}{L_1(h)} L_2(h) = (\frac{L_2(h)}{L_1(h)}) L_1(a)$ for any $a \in H_r^1$. This implies (4.13). Finally, we test (4.13) by $\bar{u} = u \geqslant 0$ to arrive at

$$I_M = \int_{\mathbb{R}^d} |\nabla u|^2 + |u|^2 \, \mathrm{d}x \, = \lambda \int_{\mathbb{R}^d} u^{p+1} \, \mathrm{d}x \, = \lambda M,$$

which implies $\lambda = I_M/M > 0$. □

4.1.3 Smoothness and decay property

We take $v = \lambda^{\frac{1}{p-1}} u$ in (4.13) such that v satisfies the following renormalised equation with $1 < p < 2^* - 1$

$$\Delta v - v + v^p = 0, \quad v \geqslant 0, \quad v \in H_r^1. \tag{4.14}$$

Proposition 4.4 (Smoothness and decay)**.** *Let $v(x) = \tilde{v}(r)$ solves (4.14), then $v \in W^{3,q}(\mathbb{R}^d)$, $\forall q \in [2, \infty)$ is a classical solution of (4.14), such that*

$$v \in C^2(\mathbb{R}^d), \quad |D^\beta v(x)| \to 0 \text{ as } |x| \to \infty, \, \forall |\beta| \leqslant 2,$$
$$\exists \varepsilon_0 > 0 \text{ s.t. } e^{\varepsilon_0 |x|}(|v| + |\nabla v|) \in L^\infty(\mathbb{R}^d).$$

Furthermore, if $\tilde{v}(r) \neq 0$, then $\tilde{v}$ solves the following ODE

$$\tilde{v}'' + \frac{d-1}{r} \tilde{v}' = \tilde{v} - \tilde{v}^p, \quad \tilde{v}(0) = a, \, \tilde{v}'(0) = 0, \tag{4.15}$$

for some real number $a > 0$.

Proof. **Step 1 Regularity by iteration**
We will use freely the following fact which we admit here without proof: If $v \in L^q(\mathbb{R}^d)$, $1 < q < \infty$, then $(1-\Delta)^{-1}v \in W^{2,q}(\mathbb{R}^d)$. (Hörmander's multiplier theorem).
By view of $v \in H^1_r(\mathbb{R}^d) \hookrightarrow L^{q_0}(\mathbb{R}^d)$, $q_0 = p+1$ and the Sobolev embedding

$$W^{2,\frac{q_j}{p}}(\mathbb{R}^d) \hookrightarrow \begin{cases} L^{q_{j+1}}(\mathbb{R}^d) \text{ if } 2 - \frac{dp}{q_j} = -\frac{d}{q_{j+1}} < 0 \\ L^q(\mathbb{R}^d),\ \forall q \in [\frac{q_j}{p}, \infty),\ \text{if } 2 - \frac{dp}{q_j} = 0 \\ L^\infty(\mathbb{R}^d) \text{ if } 2 - \frac{dp}{q_j} > 0. \end{cases}$$

we have $v \in L^\infty(\mathbb{R}^d)$. [6] Indeed, as $v^p \in L^{\frac{q_0}{p}}$, $v = (1-\Delta)^{-1}v \in W^{2,\frac{q_0}{p}}$,

- if $2 - \frac{dp}{q_0} > 0$, then by Sobolev embedding $v \in W^{2,\frac{q_0}{p}}(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$.
- if $2 - \frac{dp}{q_0} < 0$, then by Sobolev embedding $v \in W^{2,\frac{q_0}{p}}(\mathbb{R}^d) \hookrightarrow L^{q_1}(\mathbb{R}^d)$ and hence $v \in W^{2,\frac{q_1}{p}}(\mathbb{R}^d)$. If $2 - \frac{dp}{q_1} > 0$, then we are done. If not, we can continue the procedure such that there exists k with $2 - \frac{dp}{q_k} \geqslant 0$ and $2 - \frac{dp}{q_{k-1}} < 0$: This is possible since

$$\frac{1}{q_{j+1}} = -\frac{2}{d} + \frac{p}{q_j}$$
$$\Rightarrow \frac{1}{q_{j+1}} - \frac{1}{q_j} = p^j\big(\frac{p-1}{q_0} - \frac{2}{d}\big),\ \text{with } \frac{p-1}{p+1} - \frac{2}{d} < 0 \text{ if } p < 2^* - 1.$$

- if $2 - \frac{dp}{q_k} = 0$ for some $k \in \mathbb{N}$, then $v \in L^q(\mathbb{R}^d)$ for any $q \in [2, \infty)$ and we choose $q \gg 1$ such that $2 - \frac{dp}{q} > 0$.

Therefore $v^p \in L^{\frac{p+1}{p}}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and hence $v \in W^{2,q}(\mathbb{R}^d)$ for all $q \in [2,\infty)$ and thus $v^p \in W^{1,q}(\mathbb{R}^d)$ for all $q \in [2,\infty)$ which implies correspondingly $\nabla v \in W^{2,q}(\mathbb{R}^d)$ for all $q \in [2,\infty)$. Thus $v \in W^{3,q}(\mathbb{R}^d)$ for all $q \in [2,\infty)$, which implies

- $v \in C^{2,\alpha}(\mathbb{R}^d)$ for any $\alpha \in (0,1)$, by Sobolev embedding;
- $\forall |\beta| \leqslant 2$, $D^\beta v \in H^1_r(\mathbb{R}^d)$ and hence $|D^\beta v(x)| \to 0$ as $|x| \to \infty$.

[08.07.2022]

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[6] By Lemma 4.1, we have $v \in L^\infty(\mathbb{R}^d \setminus B_\varepsilon(0))$ for any positive $\varepsilon > 0$, but we don't have a priori $v \in L^\infty(\mathbb{R}^d)$.

Step 2 Decay property
Let $\theta_\varepsilon = e^{\frac{|x|}{1+\varepsilon|x|}}$, $\varepsilon > 0$ be a bounded, Lipschitz continuous function with $|\nabla\theta_\varepsilon|^2 \leqslant \theta_\varepsilon^2$, a.e. We test the equation (4.14) by $\theta_\varepsilon v$ to get (**Exercise**. Hint: Take use of the above decay property at infinity for H_r^1-functions to control v^{p+1} by v^2 outside a big ball.)

$$\int_{\mathbb{R}^d} \theta_\varepsilon v^2 \,\mathrm{d}x \leqslant C < \infty, \text{ uniformly for } \varepsilon > 0,$$

which implies $\int_{\mathbb{R}^d} e^{|x|} v^2 \,\mathrm{d}x < \infty$ as $\varepsilon \to 0$. Since v is globally Lipschitz continuous, $e^{|x|} v^{d+2}$ is uniformly bounded. Similarly we apply ∂_{x_j} to the equation (4.14) and test it by $\theta_\varepsilon \partial_{x_j} v$ to arrive at $\int_{\mathbb{R}^d} e^{|x|} |\nabla v|^2 \,\mathrm{d}x < \infty$.
Step 3 ODE equation
Hence the equation (4.14) for $v(x)$ implies the equation (4.15) for $\tilde{v}(r)$ in the classical sense, and furthermore, $\tilde{v}' = \frac{r}{d-1}(\tilde{v} - \tilde{v}^p - \tilde{v}'')$ is uniformly bounded such that $\tilde{v}'(r) \to 0$ as $r \to 0$ and thus $\tilde{v}(0) = a > 0$ since if $a = 0$ then $\tilde{v} = 0$.

□

Remark 4.2. *It is easy to show the regularity away from the origin by Lemma 4.1. Indeed, let $w = \chi v$, where $\chi \in C_0^\infty$ is a radial function with the compact support away from zero and v satisfies (4.14). Then w satisfies*

$$\Delta w - w = f, \quad f = -\chi v^p + 2\nabla\chi \cdot \nabla v + v\Delta\chi.$$

Since $v \in L^\infty$ on $\operatorname{Supp}\chi$ by virtue of (4.11), $f \in L^2(\mathbb{R}^d)$ and hence $\hat{w}(\xi) = -\frac{\hat{f}}{1+|\xi|^2}$, that is, $w \in H_r^2(\mathbb{R}^d)$. Thus $\partial_r \tilde{w} \in C^{\frac{1}{2}}((0,\infty))$ by Lemma 4.1 and $w(x) = \tilde{w}(r) \in C^1(\mathbb{R}^d\backslash\{0\})$, $v(x) \in C^1(\mathbb{R}^d\backslash\{0\})$. Now consider the equation for $\partial_r \tilde{w}$:

$$(\Delta - 1)\partial_r \tilde{w} = \partial_r f \in L^2(\mathbb{R}^d),$$

and the same argument as before implies $v(x) \in C^2(\mathbb{R}^d\backslash\{0\})$.

4.1.4 Classification of minimisers

We have the following uniqueness result for the nonnegative solution which decays at infinity of the equation (4.15) proved by [Kwong 1987]:

Proposition 4.5 (Uniqueness)**.** *There exists a unique $a > 0$ such that the solution $\tilde{v}$ of the ODE (4.15) satisfying*

$$\tilde{v}(r) \geqslant 0, \forall r \geqslant 0 \text{ and } \tilde{v}(r) \to 0 \text{ as } r \to \infty.$$

Furthermore, $\tilde{v}(r) > 0$ for all $r \geqslant 0$ and we denote the solution to be $Q(r)$: the fundamental solution of (4.15).

We do not give a proof here and interested readers can refer to Appendix B of Tao's book.
To conclude, if $u \in \mathcal{A}_M$ be a minimiser of (4.12), then $|u| \in \mathcal{A}_M$ is also a minimiser by Lemma 4.2. Thus by Propositions 4.3 and 4.4, the nonnegative function $v = \lambda^{\frac{1}{p-1}}|u| \in H_r^1$ satisfies (4.14) and the nonnegative function $\tilde{v}(r) = v(x)$ satisfies (4.15) and $\tilde{v}(r) \to 0$ as $r \to \infty$. Hence by Proposition 4.5, $v(x) = \tilde{v}(r) = Q(r) > 0$ and thus $|u| > 0$. Since $u, |u| > 0$ are two minimisers such that $\|\nabla|u|\|_{L^2(\mathbb{R}^d)} = \|\nabla u\|_{L^2(\mathbb{R}^d)}$, there exists $\gamma \in \mathbb{R}$ such that

$$u = |u|e^{i\gamma} = \lambda^{-\frac{1}{p-1}} v e^{i\gamma} = (\frac{I_M}{M})^{-\frac{1}{p-1}} Q(r) e^{i\gamma}.$$

Therefore we have obtained

Theorem 4.1 (Classification of minimisers)**.** *Let $M > 0$, $d \geqslant 2$, $1 < p < 2^* - 1$, then the minimisation problem* (4.12)

$$I_M = \inf_{u \in \mathcal{A}_M} \{\|u\|_{H^1}^2\}, \quad \mathcal{A}_M = \{u \in H_r^1(\mathbb{R}^d) \mid \int_{\mathbb{R}^d} |u|^{p+1} \, \mathrm{d}x = M\}$$

has a family of minimisers

$$e^{i\gamma}(\frac{M}{I_M})^{\frac{1}{p-1}} Q(r), \quad \gamma \in \mathbb{R},$$

where $Q > 0$ is the unique fundamental state of the equation (4.14).

Remark 4.3. *A similar proof in [Weinstein '1983 CMP] explains the optimal constant in the Gagliardo-Nirenberg's inequality (recalling Remark 2.10)*

$$\inf_{f \in H^1} \frac{\|f\|_{L^2(\mathbb{R}^d)}^{\frac{4}{d}} \|\nabla f\|_{L^2(\mathbb{R}^d)}^2}{\|f\|_{L^{2+\frac{4}{d}}(\mathbb{R}^d)}^{2+\frac{4}{d}}} = \frac{\|Q\|_{L^2(\mathbb{R}^d)}^{\frac{4}{d}}}{1 + \frac{2}{d}}.$$

4.2 Concentration compactness lemma

We give the following concentration compactness lemma, the idea of which can be used repeatedly to derive the decomposition file of a bounded sequence in H^1 (see e.g. (4.16) below). We have already used the profile decomposition techniques in the sketchy proof L^2-GWP in L^2-critical case (see (2.45)).

Lemma 4.3 (Concentration-Compactness)**.** *Let $(u_n)_{n\geqslant 1}$ be a bounded sequence in $H^1(\mathbb{R}^d)$ with $\|u_n\|^2_{L^2(\mathbb{R}^d)} = M > 0$. Then there exists a subsequence (u_{n_k}) such that the following convergence result holds true:*

$$\lim_{R\to\infty} \lim_{k\to\infty} \sup{}_{y\in\mathbb{R}^d} \int_{B(y,R)} |u_{n_k}|^2 \,\mathrm{d}x \; = m \in [0, M],$$

and in particular

(i) If $m = M$ (compactness), there exists a sequence (y_k) in $\mathbb{R}^d$ such that

$$\forall q \in [2, 2^*), \quad u_{n_k}(\cdot - y_k) \to u \text{ in } L^q(\mathbb{R}^d) \text{ as } k \to \infty;$$

(ii) If $m = 0$ (evanescence), then $\forall q \in (2, 2^*)$, $u_{n_k} \to 0$ in $L^q(\mathbb{R}^d)$ as $k \to \infty$;

(iii) If $m = \alpha M$ for some $\alpha \in (0, 1)$ (dichotomy), then there exist two bounded sequences (v_k), (w_k) with compact support in $H^1(\mathbb{R}^d)$ and $\alpha \in (0, 1)$, such that

$$\begin{aligned}
&\operatorname{Supp} v_k \cap \operatorname{Supp} w_k = \{\}, \quad d(\operatorname{Supp} v_k, \operatorname{Supp} w_k) \to \infty \text{ as } k \to \infty,\\
&\|v_k\|^2_{L^2(\mathbb{R}^d)} \to \alpha M, \quad \|w_k\|^2_{L^2(\mathbb{R}^d)} \to (1-\alpha)M, \text{ as } k \to \infty,\\
&\forall q \in [2, 2^*), \quad u_{n_k} - v_k - w_k \to 0 \text{ in } L^q(\mathbb{R}^d), \text{ as } k \to \infty,\\
&\liminf_{k\to\infty}(\|\nabla u_{n_k}\|^2_{L^2} - \|\nabla v_k\|^2_{L^2} - \|\nabla w_k\|^2_{L^2}) \geqslant 0.
\end{aligned}$$

Proof. **Step 1 Concentration functions**
Let $\rho_n : [0, \infty) \mapsto [0, M]$ be the concentration function of u_n:

$$\rho_n(R) = \sup{}_{y\in\mathbb{R}^d} \int_{B(y,R)} |u_n(x)|^2 \,\mathrm{d}x\,,$$

with the following properties:

- Monotonicity: $\forall n$, $\rho_n(R)$ increases to M as R increases to ∞;
- Concentration point: $\forall R$, the map $y \mapsto \int_{B(y,R)} |u|^2$ is continuous and tends to zero as $|y| \to \infty$, and hence the concentration point exists:

$$\forall R > 0, \quad \forall n \geqslant 0, \quad \exists y_n = y_n(R) \in \mathbb{R}^d \text{ s.t. } \rho_n(R) = \int_{B(y_n,R)} |u_n|^2 \,\mathrm{d}x\,;$$

- Uniform Hölder continuity: There exist $C, \beta > 0$ (independent on n) such that

$$\forall R_1, R_2 > 0, \quad \forall n \geqslant 0, \quad |\rho_n(R_1) - \rho_n(R_2)| \leqslant C|R_2^d - R_1^d|^\beta.$$

Indeed, suppose without loss of generality $R_1 \leqslant R_2$, then

$$
\begin{aligned}
|\rho_n(R_1) - \rho_n(R_2)| &= \int_{B(y_n^2,R_2)} |u_n|^2 \,\mathrm{d}x - \int_{B(y_n^1,R_1)} |u_n|^2 \,\mathrm{d}x \\
&= \Big(\int_{B(y_n^2,R_2)} - \int_{B(y_n^2,R_1)}\Big)|u_n|^2 \,\mathrm{d}x + \Big(\int_{B(y_n^2,R_1)} - \int_{B(y_n^1,R_1)}\Big)|u_n|^2 \,\mathrm{d}x \\
&\leqslant \int_{R_1 \leqslant |x-y_n^2| \leqslant R_2} |u_n|^2 \,\mathrm{d}x \leqslant C\Big(\int_{R_1 \leqslant |x-y_n^2| \leqslant R_2} |u_n|^{2^*} \,\mathrm{d}x\Big)^{\frac{2}{2^*}} (R_2^d - R_1^d)^{1-\frac{2}{2^*}} \\
&\leqslant C\|u_n\|^2_{H^1(\mathbb{R}^d)} (R_2^d - R_1^d)^{1-\frac{2}{2^*}} \text{ if } 2^* < \infty \text{ i.e. } d \geqslant 3 \text{ s.t. } H^1(\mathbb{R}^d) \hookrightarrow L^{2^*}(\mathbb{R}^d).
\end{aligned}
$$

By Sobolev embedding $H^1(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d)$ for all $p \in [2,\infty)$ if $d = 1, 2$, then the above argument also holds with 2^* replaced by any $p > 2$.

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By Arzela-Ascoli's Theorem, the uniform Hölder continuity of the sequence (ρ_n) above implies the existence of a subsequence (ρ_{n_k}) and a Hölder continuous monotone function $\rho(R)$ such that

$$\forall R > 0, \quad \lim_{k\to\infty} \rho_{n_k}(R) = \rho(R).$$

Let $m = \lim_{R\to\infty} \rho(R) \leqslant M$. Then (**Exercise**) there exists a sequence $R_k \to \infty$ such that

$$m = \lim_{k\to\infty} \rho_{n_k}(R_k) = \lim_{k\to\infty} \rho_{n_k}(\frac{R_k}{2}) = \lim_{R\to\infty} \rho(R).$$

Step 2 Case $m = 0$: Evanescence

Since $\rho : [0,\infty) \mapsto [0,m]$ is an increasing function, then $\rho = 0$ if $m = 0$. In particular

$$\lim_{k\to\infty} \rho_{n_k}(1) = \rho(1) = 0 = \lim_{k\to\infty} \sup{}_{y\in\mathbb{R}^d} \int_{B(y,1)} |u_{n_k}|^2 \,\mathrm{d}x\,.$$

This uniformly local strong convergence in $L^2(\mathbb{R}^d)$ implies the strong convergence in $L^q(\mathbb{R}^d)$, $q \in (2, 2^*)$: $u_{n_k} \to 0$ in $L^q(\mathbb{R}^d)$. Indeed, by use of the unity partition (Q_j) (such that each Q_j is contained in a ball of radius 1), we have the following version of Gagliardo-Nirenberg's inequality:

$$\int_{\mathbb{R}^d} |u|^{2+\frac{4}{d}} \,\mathrm{d}x = \sum_{j\geqslant 1} \|u\|^{2+\frac{4}{d}}_{L^{2+\frac{4}{d}}(Q_j)} \leqslant C \sum_{j\geqslant 1} \|u\|^{\frac{4}{d}}_{L^2(Q_j)} \|\nabla u\|^2_{L^2(Q_j)}$$

$$\leqslant C \sup_{j\geqslant 1}\|u\|_{L^2(Q_j)}^{\frac{4}{d}} \sum_{j\geqslant 1} \|\nabla u\|_{L^2(Q_j)}^2 \leqslant C\Big(\sup_{j\geqslant 1}\|u\|_{L^2(Q_j)}^2\Big)^{\frac{2}{d}} \|\nabla u\|_{L^2}^2,$$

for $d \geqslant 3$, and for $d = 1, 2$, we can take use of $\|u\|_{L^4} \leqslant C\|u\|_{L^2}^{\frac{1}{2}}\|u\|_{H^1}^{\frac{1}{2}}$. We arrive at $u_{n_k} \to 0$ in $L^{2+\frac{4}{d}}(\mathbb{R}^d)$ or $L^4(\mathbb{R}^d)$ and the interpolation in the Lebesgue spaces implies $u_{n_k} \to 0$ in L^q, $q \in (2, 2^*)$.

Step 3 Case $m = M$: Compactness

For any $R > 0$, let $y_k(R)$ be such that $\rho_{n_k}(R) = \int_{B(y_k(R),R)} |u_{n_k}|^2 \,\mathrm{d}x$. There exist R_0, k_0 such that

$$\rho_{n_k}(R_0) = \int_{B(y_k(R_0),R_0)} |u_{n_k}|^2 \,\mathrm{d}x > \frac{M}{2}, \quad \forall k \geqslant k_0,$$

and for any $\varepsilon > 0$, then there exist $R_\varepsilon, k_\varepsilon \geqslant k_0$ such that

$$\rho_{n_k}(R_\varepsilon) = \int_{B(y_k(R_\varepsilon),R_\varepsilon)} |u_{n_k}|^2 \,\mathrm{d}x > M - \varepsilon, \quad \forall k \geqslant k_\varepsilon.$$

Since u_{n_k} has the total mass M, the two balls $B(y_k(R_0), R_0) \cap B(y_k(R_\varepsilon), R_\varepsilon) \neq \{\}$ and hence there exists $R_{0\varepsilon}$ such that

$$\int_{B(y_k(R_0),R_{0\varepsilon})} |u_{n_k}|^2 \,\mathrm{d}x > M - \varepsilon, \quad \forall k \geqslant k_\varepsilon.$$

We may assume that the above holds true for all k by choosing a possibly larger $R_{0\varepsilon}$ and hence $v_k = u_{n_k}(\cdot - y_k(R_0))$ satisfies

$$\forall \varepsilon > 0, \quad \exists R_{0\varepsilon} \text{ s.t. } \forall k \geqslant 1, \quad \int_{|x|\geqslant R_{0\varepsilon}} |v_k|^2 \,\mathrm{d}x < \varepsilon.$$

By virtue of the compact embedding $H^1(\mathbb{R}^d) \hookrightarrow\hookrightarrow L^2(B(0, R_{0\varepsilon}))$, $v_k \to u$ in $L^q(\mathbb{R}^d)$, $q \in [2, 2^*)$.

Step 4 Case $0 < m < M$: Dichotomy

Intuitively we decompose u_{n_k} as

$$\begin{aligned} u_{n_k} &= u_{n_k}\mathbf{1}_{|y-y_k(\frac{R_k}{2})|\leqslant\frac{R_k}{2}} + u_{n_k}\mathbf{1}_{|y-y_k(\frac{R_k}{2})|\geqslant R_k} + u_{n_k}\mathbf{1}_{\frac{R_k}{2}<|y-y_k(\frac{R_k}{2})|<R_k} \\ &=: v_k + w_k + z_k, \end{aligned}$$

then

$$\int_{\mathbb{R}^d} |z_k|^2 \,\mathrm{d}x = \Big(\int_{B(y_k(\frac{R_k}{2}),R_k)} - \int_{B(y_k(\frac{R_k}{2}),\frac{R_k}{2})}\Big)|u_{n_k}|^2 \,\mathrm{d}x$$

$$\leqslant \rho_{n_k}(R_k) - \rho_{n_k}(\frac{R_k}{2}) \to 0 \text{ as } k \to \infty.$$

We then replace the characterised functions $\mathbf{1}_{|y-y_k(\frac{R_k}{2})|\leqslant \frac{R_k}{2}}, \mathbf{1}_{|y-y_k(\frac{R_k}{2})|\geqslant R_k}$ by regular cutoff functions θ_k, φ_k with compact supports and $\sup_k \|\nabla\theta_k\|_{L^\infty}, \sup_k \|\nabla\varphi_k\|_{L^\infty} \leqslant 4R_k^{-1}$ such that v_k, w_k are compactly supported functions with $\|v_k\|_{L^2}^2 \to m$, $\|w_k\|_{L^2}^2 \to M - m$. Since

$$\begin{aligned}|\nabla u_{n_k}|^2 - |\nabla v_k|^2 - |\nabla w_k|^2 &= |\nabla u_{n_k}|^2(1 - |\theta_k|^2 - |\varphi_k|^2) \\ &\quad - |u_{n_k}|^2(|\nabla\theta_k|^2 + |\nabla\varphi_k|^2) - \operatorname{Re}(\overline{u_{n_k}}\nabla u_{n_k}) \cdot \nabla(\theta_k^2 + \varphi_k^2) \\ &\geqslant -16|u_{n_k}|^2(R_k)^{-2} - 8|u_{n_k}||\nabla u_{n_k}|(R_k)^{-1},\end{aligned}$$

we have $\liminf_{k\to\infty} \int_{\mathbb{R}^d} |\nabla u_{n_k}|^2 - |\nabla v_k|^2 - |\nabla w_k|^2 \,\mathrm{d}x \geqslant 0$. By virtue of the uniform bound of $\|z_k\|_{H^1}$:

$$\|\nabla z_k\|_{L^2} \leqslant \|\nabla u_{n_k}\|_{L^2} + \|u_{n_k}\|_{L^2} 8\sqrt{d} R_k^{-1},$$

we have $z_k \to 0$ in $L^q(\mathbb{R}^d)$, $q \in [2, 2^*)$, □

Exercise. Give three examples of bounded H^1 sequences such that compactness/evanescence/dichotomy hold respectively.

Remark 4.4. *We can repeat the idea of the lemma (established by P.-L. Lions 1983' and we follow the proof in Cazenave 2004') to derive the decomposition profile of a sequence (u_n) in $H^s(\mathbb{R}^d)$ established by Bahouri-Gérard 1996'/P. Gérard 1998', which describes the defect of the compactness of Sobolev embeddings up to extraction:*
Let (u_n) be a bounded sequence of $H^s(\mathbb{R}^d)$, $0 < s < \frac{d}{2}$ and $\frac{d}{2} - s = \frac{d}{q}$. Then there exist a sequence of scales and cores $(\lambda_n^{(j)}, x_n^{(j)})_{(j,n)\in\mathbb{N}^2}$ in the sense that

$$j \neq k \Rightarrow \text{ either } \lim_{n\to\infty}\Big|\log(\frac{\lambda_n^{(j)}}{\lambda_n^{(k)}})\Big| = \infty \text{ or } \lim_{n\to\infty} \frac{|x_n^{(j)} - x_n^{(k)}|}{\lambda_n^{(j)}} = \infty,$$

a sequence (φ_j) in $H^s(\mathbb{R}^d)$ and a sequence $(r_n^{(j)})$ of functions such that

$$\forall J \in \mathbb{N}, \quad u_{\phi(n)}(x) = \sum_{j=0}^{J} \frac{1}{(\lambda_n^{(j)})^{\frac{d}{2}-s}} \varphi_j\Big(\frac{x - x_n^{(j)}}{\lambda_n^{(j)}}\Big) + r_n^{(J)}(x), \tag{4.16}$$

$$\lim_{J\to\infty} \limsup_{n\to\infty} \|r_n^{(J)}\|_{L^q} = 0,$$

$$\forall J \in \mathbb{N}, \quad \lim_{n\to\infty}\Big(\|u_{\phi(n)}\|_{H^s}^2 - \sum_{j=0}^{J} \|\varphi_j\|_{H^s}^2 - \|r_n^{(J)}\|_{H^s}^2\Big) = 0.$$

Notice that if $d = 3$, $s = 1$, then $q = 2^*$ which is the critical exponent of the Sobolev embedding.

4.3 Orbital stability

4.3.1 A second minimisation problem

Theorem 4.2. *Let $M > 0$, $1 < p < 1 + \frac{4}{d}$ i.e. $s_c = \frac{d}{2} - \frac{2}{p-1} < 0$ and let $Q = Q(x)$ be the fundamental state in Theorem 4.1. Then the minimisation problem*

$$J_M = \inf\{E(u) \mid u \in H^1(\mathbb{R}^d),\ \|u\|^2_{L^2(\mathbb{R}^d)} = M\} \tag{4.17}$$

is achieved by the following family of functions

$$Q_\mu(x - x_0)e^{i\gamma_0}, \quad x_0 \in \mathbb{R}^d, \quad \gamma_0 \in \mathbb{R},$$

where $E(u) = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla u|^2\,\mathrm{d}x - \frac{1}{p+1}\int_{\mathbb{R}^d} |u|^{p+1}\,\mathrm{d}x$ is the energy functional defined in (1.11), $Q_\mu = \mu^{\frac{2}{p-1}}Q(\mu x)$ and $\mu = \mu(M) = (\frac{M}{\|Q\|^2_{L^2}})^{-\frac{1}{2s_c}}$. Furthermore, all the minimising sequences are relatively compact in $H^1(\mathbb{R}^d)$ up to translation and rotation: For the sequence (u_n) in $H^1(\mathbb{R}^d)$ such that

$$\|u_n\|^2_{L^2} \to M, \quad E(u_n) \to J_M,$$

there exist $(x_n) \subset \mathbb{R}^d$, $(\gamma_n) \subset \mathbb{R}$ and a subsequence $(\phi(n))$ such that

$$u_{\phi(n)}(\cdot - x_{\phi(n)})e^{i\gamma_{\phi(n)}} \to Q_\mu \text{ in } H^1(\mathbb{R}^d).$$

Sketchy proof. **Step 1 Properties of J_M, $M > 0$**
We have the following properties for J_M:

- J_M has a lower bound: $J_M > -\infty$.

  Indeed, recall the Gagliardo-Nirenberg's inequality in Corollary 2.1:

$$\|u\|^{p+1}_{L^{p+1}(\mathbb{R}^d)} \leqslant C\|\nabla u\|^{2a}_{L^2(\mathbb{R}^d)}\|u\|^{2b}_{L^2(\mathbb{R}^d)}, \tag{4.18}$$
$$a = \frac{d(p-1)}{4}, \quad b = \frac{d+2}{4} - \frac{(d-2)p}{4}.$$

  We hence have

$$E(u) \geqslant \frac{1}{2}\|\nabla u\|^2_{L^2(\mathbb{R}^d)} - C\|\nabla u\|^{2a}_{L^2(\mathbb{R}^d)}\|u\|^{2b}_{L^2(\mathbb{R}^d)},$$
$$\text{with } a < 1 \text{ since } p < 1 + \frac{4}{d}.$$

  As $\|u\|^2_{L^2(\mathbb{R}^d)} = M$, we derive $J_M > -\infty$ by Young's inequality.

- J_M has a negative upper bound: $J_M < 0$.

  Indeed, fix some $u \in H^1(\mathbb{R}^d)$ with $\|u\|^2_{L^2(\mathbb{R}^d)} = M$. Then the rescaled function $u^\lambda(x) = \lambda^{\frac{d}{2}} u(\lambda x)$, $\lambda > 0$ satisfies $\|u^\lambda\|^2_{L^2(\mathbb{R}^d)} = \|u\|^2_{L^2(\mathbb{R}^d)} = M$ and

  $$E(u^\lambda) = \lambda^2 \Big[\frac{1}{2} \int_{\mathbb{R}^d} |\nabla u|^2 \,\mathrm{d}x - \frac{1}{(p+1)\lambda^{(p-1)|s_c|}} \int_{\mathbb{R}^d} |u|^{p+1} \,\mathrm{d}x \Big],$$

  and hence $E(u^\lambda) < 0$ for $\lambda > 0$ small enough.

- J_M is homogeneous in M: $J_M = M^{\frac{1-s_c}{|s_c|}} J_1$.

  Indeed, for any $u \in H^1(\mathbb{R}^d)$, the rescaled function $u_\lambda(x) = \lambda^{\frac{2}{p-1}} u(\lambda x)$, $\lambda > 0$ satisfies $\|u_\lambda\|^2_{L^2(\mathbb{R}^d)} = \lambda^{-2s_c} \|u\|^2_{L^2(\mathbb{R}^d)} = \lambda^{-2s_c} M$ and

  $$E(u_\lambda) = \lambda^{2(1-s_c)} E(u),$$

  and hence $J_{\lambda^{-2s_c} M} = \lambda^{2(1-s_c)} J_M$ and we can choose in particular $\lambda = M^{\frac{1}{2s_c}}$.

Step 2 Existence of the minimiser

Let (u_n) be a minimizing sequence and we are going to show its compactness (up to extraction of subsequence and translation) by Lemma 4.3. We first have the following facts:

- Any subsequence is not evanescent.

  Indeed, suppose by contradiction that $u_{n_k} \to 0$ in $L^{p+1}(\mathbb{R}^d)$ as $k \to \infty$, then

  $$\begin{aligned} J_M = \lim_{k\to\infty} E(u_{n_k}) &= \lim_{k\to\infty} \Big[\frac{1}{2} \int_{\mathbb{R}^d} |\nabla u_{n_k}|^2 \,\mathrm{d}x - \frac{1}{p+1} \int_{\mathbb{R}^d} |u_{n_k}|^{p+1} \,\mathrm{d}x \Big] \\ &= \lim_{k\to\infty} \frac{1}{2} \int_{\mathbb{R}^d} |\nabla u_{n_k}|^2 \,\mathrm{d}x \geqslant 0, \end{aligned}$$

  which is in contradiction with $J_M < 0$.

- Any subsequence is not dichotomous.

  Indeed, suppose by contraction that there exist two sequences v_{n_k}, w_{n_k} with disadjoint supports such that

  $$\int_{\mathbb{R}^d} |v_{n_k}|^2 \,\mathrm{d}x \to \alpha M, \quad \int_{\mathbb{R}^d} |w_{n_k}|^2 \,\mathrm{d}x \to (1-\alpha) M, \quad \alpha \in (0,1),$$
  $$\|u_{n_k}\|^{p+1}_{L^{p+1}} - \|v_{n_k}\|^{p+1}_{L^{p+1}} - \|w_{n_k}\|^{p+1}_{L^{p+1}} \to 0,$$

$$\liminf_{k\to\infty}[\|\nabla u_{n_k}\|_{L^2}^2 - \|\nabla v_{n_k}\|_{L^2}^2 - \|\nabla w_{n_k}\|_{L^2}^2] \geqslant 0,$$

then

$$J_M = \lim_{k\to\infty} E(u_{n_k}) \geqslant \limsup_{k\to\infty}[E(v_{n_k}) + E(w_{n_k})] \geqslant J_{\alpha M} + J_{(1-\alpha)M}.$$

By the homogeneity property of J_M and $J_1 < 0$ we have

$$1 \leqslant \alpha^{\frac{1-s_c}{|s_c|}} + (1-\alpha)^{\frac{1-s_c}{|s_c|}}, \text{ with } \frac{1-s_c}{|s_c|} > 1,$$

which is an contradiction with the fact that if $\theta > 1$, then $f(\alpha) := \alpha^\theta + (1-\alpha)^\theta < f(0) = f(1) = 1$ for all $\alpha \in (0,1)$.

Hence by Lemma 4.3, there exist $(x_k) \subset \mathbb{R}^d$ such that $u_{n_k}(x - x_k) \to u$ in $L^2(\mathbb{R}^d) \cap L^{p+1}(\mathbb{R}^d)$, and hence

$$J_M = \lim_{k\to\infty} E(u_{n_k}) = \lim_{k\to\infty} E(u_{n_k}(x - x_k)) \geqslant E(u).$$

Therefore $u \in H^1(\mathbb{R}^d)$ is a minimiser and $\lim_{k\to\infty} E(u_{n_k}) = E(u)$, $u_{n_k}(x - x_k) \to u$ in $H^1(\mathbb{R}^d)$.

Step 3 Classification of the minimisers

We follow the strategy in Subsection 4.1 to classify the minimisers of the minimisation problem (4.17):

- If u is a minimiser of (4.17), then by Lemma 4.2, we know that $|u| \geqslant 0$ is also a minimiser of (4.17).
- If $u \geqslant 0$ is a minimiser of (4.17), then (**Exercise**) we follow the idea in the proof of Proposition 4.3 to derive the existence of $\tilde{\mu} = \tilde{\mu}(M) \in \mathbb{R}$ (independent of the minimisers) such that

$$\Delta u + u^p = \tilde{\mu} u, \quad u \geqslant 0, \quad u \in H^1(\mathbb{R}^d). \tag{4.19}$$

Hence (**Exercise**) we derive (with the notations a, b given in (4.18))

$$\frac{1}{2}\int_{\mathbb{R}^d} |\nabla u|^2 \,\mathrm{d}x = \frac{a}{p+1}\int_{\mathbb{R}^d} u^{p+1} \,\mathrm{d}x\,,$$
$$\tilde{\mu}\frac{1}{2}\int_{\mathbb{R}^d} |u|^2 = \frac{b}{p+1}\int_{\mathbb{R}^d} u^{p+1} \,\mathrm{d}x\,,$$

by testing (4.19) by $u \geqslant 0$ and by $(\frac{d}{2} + x \cdot \nabla)u$ respectively. Therefore

$$J_M = E(u) = \frac{a-1}{p+1}\int_{\mathbb{R}^d} u^{p+1} \,\mathrm{d}x < 0 \text{ for } 1 < p < 1 + \frac{4}{d} \text{ s.t. } a < 1,$$

and thus

$$\tilde{\mu} = \tilde{\mu}(M) = \frac{2bJ_M}{(a-1)M} = \frac{2b}{a-1} J_1 M^{-\frac{1}{s_c}} > 0.$$

– If $u \in H^1$, $u \geqslant 0$ satisfies (4.19), then the rescaled solution $\tilde{u}_{\mu^{-1}}(x) = \frac{1}{\mu^{\frac{2}{p-1}}} u(\frac{x}{\mu})$, $\mu = \sqrt{\tilde{\mu}}$ satisfies the equation (4.14)

$$\Delta v + v^p = v, \quad v \geqslant 0, \quad v \in H^1(\mathbb{R}^d). \tag{4.20}$$

We have the nontrivial symmetry result established by [Gidas, Ni and Nirenberg, 1979] which we do not prove here:
If v satisfies (4.20), then there exists $x_0 \in \mathbb{R}^d$ such that $v(x - x_0) \in H^1_r(\mathbb{R}^d)$.
Hence by Propositions 4.4 and 4.5, the solution of (4.20) is indeed unique:

$$v(x - x_0) = Q(x), \text{ for some } x_0 \in \mathbb{R}^d,$$

where $Q(x)$ is the fundamental solution in Proposition 4.5.

– Conclusion: If u is a minimiser of the minimiser problem (4.17), then there exist $(\gamma_0, x_0) \in \mathbb{R} \times \mathbb{R}^d$ such that

$$u(x) = Q_\mu(x - x_0)e^{i\gamma_0}, \text{ with } Q_\mu(x) = \mu^{\frac{2}{p-1}} Q(\mu x),$$

where

$$\mu = \mu(M) = \frac{\mu(M)}{\mu(\|Q\|_{L^2}^2)} = (\frac{\tilde{\mu}(M)}{\tilde{\mu}(\|Q\|_{L^2}^2)})^{\frac{1}{2}} = (\frac{M}{\|Q\|_{L^2}^2})^{-\frac{1}{2s_c}}.$$

□

4.3.2 Orbital stability

Recall the "natural" stability notion of the solitary waves $e^{it}Q(r)$ in H^1 for $1 < p < 1 + \frac{4}{d}$:
Let $u_0 \in H^1$ and let $u \in C(\mathbb{R}; H^1)$ be the global solution of (NLS) with the initial data u_0. Then for all $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that for all $u_0 \in H^1$:

$$\|u_0 - Q\|_{H^1} < \delta \text{ implies } \sup_{t \geqslant 0} \|u(t, x) - e^{it}Q(x)\|_{H^1} < \varepsilon.$$

This strong stability property is not suitable for (NLS) by virtue of the symmetries of the equation (NLS). Indeed the scaling invariance and the Galilean invariance in Subsection 1.1.3 supply two obvious examples of strong instability (**Exercise**):

- By scaling symmetry, for any $\lambda > 0$, there exists a solution $u_\lambda(t,x) = \lambda^{\frac{2}{p-1}} Q(\lambda x) e^{i\lambda^2 t}$ of (NLS) with the initial data $(u_0)_\lambda(x) = \lambda^{\frac{2}{p-1}} Q(\lambda x)$. We have
$$\|(u_0)_\lambda - Q\|_{H^1} \to 0 \text{ as } \lambda \to 1,$$
while for any $\lambda \neq 1$,
$$\sup{}_t \|u_\lambda(t,x) - e^{it}Q(x)\|_{H^1} \geqslant \|Q\|_{H^1}.$$

- By Galilean invariance, for any $v \in \mathbb{R}^d$, there exists a solution $u_v = e^{i(x\cdot v - |v|^2 t + t)} Q(x - 2vt)$ of (NLS) with the initial data $(u_0)_v = e^{iv\cdot x} Q(x)$. We have
$$\|(u_0)_v - Q\|_{H^1} \to 0 \text{ as } |v| \to 0,$$
while whenever $v \neq 0$,
$$\sup{}_t \|u_v(t,x) - e^{it}Q(x)\|_{H^1} \geqslant \|Q\|_{H^1}.$$

By the variational characterisation of the solitary waves in Theorem 4.2, we have the orbital stability results:

Theorem 4.3 (Orbital stability of the solitary waves). *Let $1 < p < 1 + \frac{4}{d}$, $\kappa = -1$. Then for all $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that for all $\|u_0 - Q\|_{H^1(\mathbb{R}^d)} < \delta$, there exist $(x(t), \gamma(t)) \in \mathbb{R}^d \times \mathbb{R}$ so that the corresponding solution $u \in C(\mathbb{R}; H^1)$ of* (NLS) *satisfying*
$$\sup{}_t \|u(t,x) - Q(x - x(t)) e^{i\gamma(t)}\|_{H^1(\mathbb{R}^d)} < \varepsilon.$$

Proof. Suppose by contradiction that there exist $\varepsilon_0 > 0$, a sequence of times $t_n \geqslant 0$ and a sequence of solutions $u_n \in C(\mathbb{R}; H^1)$ such that
$$\|u_n(0,x) - Q\|_{H^1} \to 0 \text{ as } n \to \infty,$$
$$\|u_n(t_n,x) - Q(x - x_0)e^{i\gamma_0}\|_{H^1} > \varepsilon_0 > 0, \quad \forall x_0 \in \mathbb{R}^d,\ \forall \gamma_0 \in \mathbb{R}\,.$$
Then we have
$$\|u_n(0,\cdot)\|_{L^2}^2 \to \|Q\|_{L^2}^2 := M, \quad E(u_n(0,x)) \to E(Q) = J_M.$$
By the mass and energy conservation laws, we have
$$\|u_n(t_n,\cdot)\|_{L^2}^2 = \|u_n(0,\cdot)\|_{L^2}^2 \to M, \quad E(u_n(t_n,x)) = E(u_n(0,x)) \to J_M,$$
and hence by Theorem 4.2, there exist $(\phi(n)) \subset \mathbb{N}$, $(x_{\phi(n)}) \subset \mathbb{R}^d$, $(\gamma_{\phi(n)}) \subset \mathbb{R}$ such that
$$u_n(t_n, x - x_{\phi(n)}) e^{i\gamma_{\phi(n)}} \to Q \text{ in } H^1(\mathbb{R}^d), \text{ as } n \to \infty,$$
which is in contradiction to the assumption. □

Remark 4.5. *If* $p \geqslant 1 + \frac{4}{d}$, $\kappa = -1$ *then the solitary wave* $u(t,x) = e^{it}Q(x)$ *is unstable in the sense that there exists* $(Q_n)_n \subset H^1(\mathbb{R}^d)$ *such that* $Q_n \to Q$ *in* $H^1(\mathbb{R}^d)$ *while the corresponding solution* $u_n(t,x)$ *blows up in finite time. Indeed, as* $\Delta Q - Q + Q^p = 0$, *if* $p = 1 + \frac{4}{d}$, *then*

$$E(Q) = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla Q|^2 \,\mathrm{d}x - \frac{1}{p+1}\int_{\mathbb{R}^d} |Q|^{p+1} \,\mathrm{d}x = 0$$

and $E(\lambda Q) < 0$ *for any* $\lambda > 1$, *and hence we have the blowup results by Theorem 3.1 for the solutions with initial data* $(1 + \frac{1}{n})Q$. *Similarly we notice that if* $p > 1 + \frac{4}{d}$, *then with* $a = \frac{d(p-1)}{4} > 1$

$$\frac{1}{2}\int_{\mathbb{R}^d} |\nabla Q|^2 \,\mathrm{d}x - \frac{a}{p+1}\int_{\mathbb{R}^d} |Q|^{p+1} \,\mathrm{d}x = 0.$$

For any $\varepsilon > 0$ *we can take* λ_ε *small enough such that* $E(Q_{\lambda_\varepsilon}) = \lambda_\varepsilon^{2(\frac{2}{p-1} - \frac{d}{2} + 1)} E(Q) < \varepsilon$ *where* $Q_\lambda = \lambda^{\frac{2}{p-1}} Q(\lambda x)$ *and hence there exists* λ *larger but close to* 1 *such that* $E(\lambda Q_{\lambda_\varepsilon}) < 0$.