Summer term 2022
Dispersive equations
Exercise sheet 1
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Exercise 1

Consider the 1 D cubic focusing Schrödinger equation

$$\begin{cases} \mathrm{i}\partial_t u + \Delta_x u = -|u|^2 u, & (t,x) \in \mathbb{R} \times \mathbb{R}, \\ u|_{t=0} = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{1}$$

Show that the function

$$u : \mathbb{R} \times \mathbb{R} \to \mathbb{C}, \quad u(t,x) = \sqrt{2}\mathrm{e}^{\mathrm{i}t} \operatorname{sech} x$$

is a solution to (1).

Exercise 2

a) Consider the linear **Schrödinger equation**

$$\begin{cases} \mathrm{i}\partial_t u + \Delta_x u = 0, & (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ u|_{t=0} = u_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{2}$$

from Section 1.1.1 in the lecture notes. Show that the solution to (2) is given by

$$u(t,x) = (4\pi \mathrm{i}t)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}\frac{|x-y|^2}{4t}} u_0(y)\,\mathrm{d}y.$$

b) Consider the linear **heat equation**

$$\begin{cases} \partial_t u - \Delta_x u = 0, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d, \\ u|_{t=0} = u_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{3}$$

from Section 1.1.1 in the lecture notes. Show that the solution to (3) is given by

$$u(t,x) = (4\pi t)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \mathrm{e}^{-\frac{|x-y|^2}{4t}} u_0(y)\,\mathrm{d}y, \quad t \geq 0.$$

Exercise 3

Let $s_c = \frac{d}{2} - \frac{2}{p-1}$. Let $u_0 \in H^s(\mathbb{R}^d)$, $s \in \mathbb{R}$ and $u_{0,\lambda}(x) = \lambda^{\frac{-2}{p-1}} u_0(\frac{x}{\lambda})$, $\lambda > 0$. Show that

$$\|u_{0,\lambda}\|_{\dot{H}^s(\mathbb{R}^d)} = \lambda^{-s+s_c} \|u_0\|_{\dot{H}^s(\mathbb{R}^d)}. \tag{4}$$

Exercise 4

Suppose that $u(t,x) \in \mathcal{S}(\mathbb{R} \times \mathbb{R}^d)$ is a solution of the Nonlinear Schrödinger equation:

$$\begin{cases} \mathrm{i}\partial_t u + \Delta_x u = \kappa |u|^{p-1} u, & (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ u|_{t=0} = u_0(x), & x \in \mathbb{R}^d. \end{cases} \tag{5}$$

where $\kappa = \pm 1$.

Show that we have the following conservation laws:

- Mass conservation law

$$M(u)(t) = \int_{\mathbb{R}^d} |u(t,x)|^2 \,\mathrm{d}x = M(u)(0),$$

- Momentum conservation law

$$P_j(u)(t) = \operatorname{Im} \int_{\mathbb{R}^d} \overline{u} \partial_{x_j} u \,\mathrm{d}x = P_j(u)(0), \quad 1 \le j \le d,$$

- Energy conservation law

$$E(u)(t) = \frac{1}{2} \int_{\mathbb{R}^d} |\nabla u|^2 \, dx + \frac{\kappa}{p+1} \int_{\mathbb{R}^d} |u|^{p+1} \,\mathrm{d}x = E(u)(0).$$

Hint: We use integration by part.

- Mass conservation law: test (5) by $\overline{u}$, then take the imaginary part.
- Momentum conservation law: test (5) by $\partial_{x_j} \overline{u}$, then take the real part.
- Energy conservation law: test (5) by $\Delta \overline{u} - \kappa |u|^{p-1} \overline{u}$, then take the imaginary part.

Exercise 5

Consider the Lagrangian $L_{\text{NLS}} : \mathbb{C} \times \mathbb{C}^{d+1} \mapsto \mathbb{R}$ defined as follows:

$$L_{\text{NLS}}(u, u_0, u_1, \cdots, u_d) = \frac{i}{2}(\bar{u} u_0 - u \overline{u_0}) - |(u_1, \cdots, u_d)|^2 - \frac{2\kappa}{p+1} |u|^{p+1}. \tag{6}$$

Show that the nonlinear Schrödinger equation (5) can be viewed as the Euler-Lagrangian equation associated to the Lagrangian L_{NLS}.

Exercise 6

Consider the scaled 1 D cubic defocusing Schrödinger equation

$$\begin{cases} \mathrm{i}\partial_t u + \Delta_x u = 2|u|^2 u, & (t,x) \in \mathbb{R} \times \mathbb{R}, \\ u|_{t=0} = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{7}$$

Consider the two systems

$$\psi_x = \begin{pmatrix} -\mathrm{i}z & u \\ \bar{u} & \mathrm{i}z \end{pmatrix} \psi =: U\psi,$$

$$\psi_t = \mathrm{i} \begin{pmatrix} -2z^2 - |u|^2 & -2\mathrm{i}zu + u_x \\ -2\mathrm{i}z\bar{u} - \bar{u}_x & 2z^2 + |u|^2 \end{pmatrix} \psi =: V\psi,$$

where z is a parameter, (t, x) are space and time variables, u is some given function and $\psi : \mathbb{R} \times \mathbb{R} \to \mathbb{C}^2$ is the unknown vector-valued function. Show that if the following compatibility condition

$$U_t = V_x + [V, U] \text{ with } [V, U] := VU - UV$$

holds then u solves (7).