Summer term 2022 **Karlsruhe Institute of Technology**
Dispersive equations JProf. Dr. Xian Liao
Exercise sheet 2 M.Sc. Julia Henninger
Due date: 13.05.2022

Recall from the lecture notes:

$$\psi_x = \begin{pmatrix} -\mathrm{i}z & u \\ \bar{u} & \mathrm{i}z \end{pmatrix} \psi =: U\psi, \tag{1}$$

$$\psi_t = \mathrm{i} \begin{pmatrix} -2z^2 - |u|^2 & -2\mathrm{i}zu + u_x \\ -2\mathrm{i}z\bar{u} - \bar{u}_x & 2z^2 + |u|^2 \end{pmatrix} \psi =: V\psi, \tag{2}$$

Exercise 1

Let $J^{\pm}$ be two fundamental solutions of the ODE (1) satisfying

$$\lim_{x\to\pm\infty} (J^{\pm}(x)\mathrm{e}^{\mathrm{i}zx\sigma_3} - \mathrm{Id}\,) = 0, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

By the ODE theory, there exists an invertible matrix $S = S(z)$ (independent of x) such that

$$J^-(x; z) = J^+(x; z)S(z)^{-1}.$$

Show that

a) S is of the form $S(z) = \begin{pmatrix} \bar{a} & -\bar{b} \\ -b & a \end{pmatrix}$.

   Hint: Use symmetry of (1), meaning if $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ solves (1), then $\begin{pmatrix} \overline{\psi_2} \\ \overline{\psi_1} \end{pmatrix}$ solves (1) as well.

b) $\det(S) = |a|^2 - |b|^2 = 1$.

Exercise 2

The matrices

$$W^{\pm}(t, x; z) = J^{\pm}(t, x; z)\mathrm{e}^{-2\mathrm{i}z^2t\sigma_3}$$

are simultaneous fundamental solutions of the ODEs (1) and (2). Show that the evolution of the scattering matrix S is described by $a(t; z) = a(0; z)$ and $b(t; z)\mathrm{e}^{4\mathrm{i}z^2t}b(0; z)$.

Exercise 3

Let $u(t, x) \in \mathcal{S}(\mathbb{R} \times \mathbb{R})$ and $T_2(z) = \int_{x_1<x_2} \mathrm{e}^{2\mathrm{i}z(x_2-x_1)}u(x_2)\overline{u(x_1)}\,dx_1\,dx_2$. Use the inverse Fourier transform $u(x) = \frac{\mathrm{i}}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}x\xi}\widehat{u}(\xi)\,d\xi$ to show

$$T_2(z) = \int_{\mathbb{R}} \frac{\mathrm{i}}{2z + \xi}\,\widehat{u}(\xi)\,\overline{\widehat{u}(\xi)}\,d\xi.$$

Exercise 4

Consider the following Hamiltonians in the NLS hierarchy

$$
\begin{aligned}
H_0 &= \int_{\mathbb{R}} |u|^2 \,\mathrm{d}x, \\
H_1 &= \frac{1}{2}\frac{1}{i}\int_{\mathbb{R}} u\partial_x \bar{u}\,\mathrm{d}x = \frac{1}{2}\,\mathrm{Im}\int_{\mathbb{R}} u\partial_x \bar{u}\,\mathrm{d}x, \\
H_2 &= \frac{1}{4}\int_{\mathbb{R}} \big(|\partial_x u|^2 + |u|^4\big)\,\mathrm{d}x, \\
H_3 &= \frac{1}{8}\,\mathrm{Im}\int_{\mathbb{R}} \big(\partial_x u \partial_{xx}\bar{u} + 3|u|^2 u\partial_x \bar{u}\big)\,\mathrm{d}x.
\end{aligned}
$$

Show that these Hamiltonians generate the corresponding Hamiltonian flows as follows:

- H_0 generates the phase shifts: $u(t) = \mathrm{e}^{-\mathrm{i}t}u_0$;
- H_1 generates the group of translations: $u(t,x) = u_0(x+2t)$;
- H_2 generates the (rescaled) defocusing cubic (NLS) flow ;
- H_3 generates the defocusing mKdV flow mKdV.

Exercise 5

Consider the so-called Madelung transform

$$
u(t, \sqrt{2}x) = \sqrt{\rho(t,x)}\mathrm{e}^{\mathrm{i}\phi(t,x)}.
$$

Show that we obtain the following system for the unknown (ρ, v) with $v = \nabla\phi$ from (NLS):

$$
\begin{cases}
\partial_t \rho + \mathrm{div}_x(\rho v) = 0, \\
\partial_t v + v\cdot\nabla_x v + \nabla_x(\kappa\rho^{\frac{p-1}{2}}) = \nabla_x\big(\frac{\Delta\sqrt{\rho}}{2\sqrt{\rho}}\big).
\end{cases}
$$