Summer term 2022
Dispersive equations
Exercise sheet 3
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Exercise 1

Consider the Laplace equation:

$$\begin{cases} v_{tt} + v_{xx} = 0, \\ v|_{t=0} = 0, \quad v_t|_{t=0} = f(x). \end{cases} \tag{1}$$

Show the existence and the uniqueness results of the solution for the above Cauchy problem with the following initial data sequence

$$(v, v_t)|_{t=0} = (0, f_n) = (0, \mathrm{e}^{-\sqrt{n}} n \sin(nx)) \to 0 \text{ as } n \to \infty, \text{ in any } C_b^k(\mathbb{R}),\ k \in \mathbb{N},$$

while the continuity of the flow map fails.

Exercise 2

Recall the Schrödinger group

$$S(t)g = \mathrm{e}^{\mathrm{i}t\Delta} g = K_t * g = (4\pi \mathrm{i}t)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}\frac{|x-y|^2}{4t}} g(y)dy, \quad t \in \mathbb{R} \tag{St}$$

and the heat semigroup

$$H(t)f = \mathrm{e}^{t\Delta} f = A_t * f = (4\pi t)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \mathrm{e}^{-\frac{|x-y|^2}{4t}} f(y)dy, \quad t > 0. \tag{Ht}$$

i) Show that the operator $S(t)$, $t > 0$ does not map

  a) from $L^2(\mathbb{R}^d)$ to $L^r(\mathbb{R}^d)$ or from $L^r(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$ for $r \neq 2$;
  b) from $L^r(\mathbb{R}^d)$ to $L^r(\mathbb{R}^d)$ for $r \neq 2$;
  c) from $L^r(\mathbb{R}^d)$ to $L^{r_1}(\mathbb{R}^d)$ for any $r > 2$;
  d) from $H^s(\mathbb{R}^d)$ to $H^{s'}(\mathbb{R}^d)$ for $s' > s$.

ii) Show that the operator $H(t)$, $t > 0$ maps

  a) from $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$ for any $q \geq p$;
  b) from $H^s(\mathbb{R}^d)$ to $H^{s'}(\mathbb{R}^d)$ for any $s' \geq s$.

Exercise 3

Show the Hardy-Littlewood-Sobolev inequality

$$\|g * |\cdot|^{-\alpha}\|_{L^q(\mathbb{R}^n)} \leq C(p, q, \alpha, n) \|g\|_{L^m(\mathbb{R}^n)},$$
$$1 + \frac{1}{q} = \frac{1}{m} + \frac{\alpha}{n}, \quad 0 < \alpha < n, \quad 1 < m < q < \infty.$$