Summary term 2022
Dispersive equations
Exercise sheet 4
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Exercise 1

Find a function in $H^1(\mathbb{R}^2)$ but not in $L^\infty(\mathbb{R}^2)$.

Exercise 2

Let $s > 0$ and

$$p_c = \begin{cases} \dfrac{2d}{d-2s}, & \text{if} \quad s < \dfrac{d}{2}, \\ \infty, & \text{if} \quad s \geq \dfrac{d}{2}. \end{cases}$$

Take the smooth mollifier function: $\varphi \in C_0^\infty(\mathbb{R}^d)$, $\varphi \geq 0$, $\int_{\mathbb{R}^d} \varphi \, dx = 1$ and its rescaled functions $\varphi_\varepsilon(x) = \varepsilon^{-d}\varphi(\varepsilon^{-1}x)$ for $\varepsilon > 0$. Show the following

a) $\sup_{\|g\|_{H^s} \leq 1} \|\varphi_\varepsilon * g - g\|_{L^2(\mathbb{R}^d)} \to 0$ as $\varepsilon \to 0$, if $s > 0$,

b) $\sup_{\|g\|_{H^s} \leq 1} \|\varphi_\varepsilon * g - g\|_{L^\infty(\mathbb{R}^d)} \to 0$ as $\varepsilon \to 0$, if $s > \frac{d}{2}$,

c) $\sup_{\|g\|_{L^2} = 1} \|\varphi_\varepsilon * g - g\|_{L^2(\mathbb{R}^d)} \geq 1$ for any $\varepsilon > 0$.

Exercise 3

Let $s > 0$ and p_c as above. Show that the embeddings $H^s(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d)$, $2 \leq p < p_c$ and $H^s(\mathbb{R}^d) \hookrightarrow L^{p_c}_{loc}(\mathbb{R}^d)$ if $s < \frac{d}{2}$ are both not compact.

Exercise 4

Let $1 < p < \infty$, $\kappa = \pm 1$, $u_0 \in H^1(\mathbb{R}^d)$ and $d = 1, 2$. Let $S(t) = e^{it\Delta}$ denote the Schrödinger group. Show that the nonlinear map

$$\Psi : u \mapsto \Psi(u) = S(t)u_0 - i\kappa \int_0^t S(t-t')\big(|u(t')|^{p-1}u(t')\big)dt' \tag{1}$$

is well-defined and contractive in

$$Y_T(R) = \{u \in C([-T,T]; H^1) \cap L^q([-T,T]; W^{1,\rho}) | \|u\|_T = \|u\|_{L^\infty_T L^2} + \|u\|_{L^q_T W^{1,\rho}} \leq R\}$$

for appropriately chosen admissible exponent pair (q, ρ) with $q > p \geq \rho/2 > 1$ and $R,\ T > 0$.