Summer term 2022 **Karlsruhe Institute of Technology**
Dispersive equations JProf. Dr. Xian Liao
Exercise sheet 5 M.Sc. Julia Henninger
Due date: 04.07.2022

Recall from the lecture notes:

- Nonlinear Schrödinger equation

$$\begin{cases} i\partial_t u + \Delta u = \kappa |u|^{p-1}u, \quad \kappa = \pm 1, \\ u|_{t=0} = u_0(x). \end{cases} \tag{NLS}$$

- Virial potential

$$V(u) = \int_{\mathbb{R}^d} |x|^2 |u|^2 \, dx, \tag{1}$$

- Morawetz action

$$W(u) = \operatorname{Im} \sum_{j=1}^{d} \int_{\mathbb{R}^d} x_j (\bar{u} \partial_{x_j} u) \, dx \equiv \operatorname{Im} \int_{\mathbb{R}^d} r(\bar{u}\partial_r u) \, dx, \quad r = |x|. \tag{2}$$

Exercise 1

Let $u(t,x) \in \mathcal{S}(\mathbb{R} \times \mathbb{R}^d)$ be a solution of (NLS). Show that

$$\frac{1}{4}\frac{d}{dt} V(u(t)) = W(u(t)), \tag{3}$$

and

$$\frac{1}{2}\frac{d}{dt} W(u(t)) = \int_{\mathbb{R}^d} |\nabla u|^2 \, dx + \kappa \big(\frac{d}{2} - \frac{d}{p+1}\big) \int_{\mathbb{R}^d} |u|^{p+1} \, dx. \tag{4}$$

Hint: Making use of the Pohozaev's Identity

$$\begin{aligned} &\operatorname{Re} \int_{\mathbb{R}^d} \Delta \bar{u} \big(x \cdot \nabla u\big) \, dx = \big(\frac{d}{2} - 1\big) \int_{\mathbb{R}^d} |\nabla u|^2 \, dx, \quad \forall u \in \mathcal{S}(\mathbb{R}^d), \\ &\text{or equivalently, } \operatorname{Re} \int_{\mathbb{R}^d} \Delta \bar{u} \big(\frac{d}{2} u + x \cdot \nabla u\big) \, dx = - \int_{\mathbb{R}^d} |\nabla u|^2 \, dx, \end{aligned} \tag{5}$$

where $|\nabla u|^2 = \sum_{j=1}^d ((\partial_{x_j} \operatorname{Re} u)^2 + (\partial_{x_j} \operatorname{Im} u)^2)$.

Exercise 2

Let $p \in (1, 2^* - 1)$ be energy subcritical exponent. Let $u_0 \in \Sigma$ and let $u \in C([-T,T]; H^1)$, $T < \infty$ be the solution of (NLS). Recall the operator $P = x + 2it\nabla$. Show that

i) $u \in C([-T,T]; \Sigma) \cap L^q([-T,T]; W^{1,\rho})$, $Pu \in L^q([-T,T]; L^\rho)$ for any admissible exponent pair (q, ρ).

ii) the mass and energy conservation laws as well as the virial and Morawetz identities hold for u on the existence time interval $[-T, T]$: For any $t \in [-T, T]$,

$$M(u(t)) = M(u_0), \quad E(u(t)) = E(u_0),$$
$$\frac{1}{4}V(u(t)) - \frac{1}{4}V(u_0) = \int_0^t W(u(t'))dt',$$
$$\frac{1}{2}W(u(t)) - \frac{1}{2}W(u_0) = \int_0^t \int_{\mathbb{R}^d} |\nabla u|^2 \, dxdt + \kappa(\frac{d}{2} - \frac{d}{p+1}) \int_0^t \int_{\mathbb{R}^d} |u|^{p+1} \, dxdt.$$

Exercise 3

Let $d = 1, 2$, $1 + \frac{4}{d} \leq p < \infty$ and $u_0 \in H^1(\mathbb{R}^d)$. Let $u(t,x)$ be the solution of the Cauchy problem (NLS) satisfying $\lim_{t \uparrow T^*} \|\nabla u(t)\|_{L^2(\mathbb{R}^d)} = +\infty$, then

$$\|\nabla u(t)\|_{L^2(\mathbb{R}^d)} \geq C_0 (T^* - t)^{-\frac{1}{p-1}}, \quad \forall t \in [0, T^*).$$

Hint: Recall the H^1 local well-posedness results for $d = 1, 2$. For any time $t_0 < T^*$ with $\|u(t_0)\|_{H^1(\mathbb{R}^d)} < \infty$, the solution u with $\|u\|_{L^\infty([t_0, t_0+T]; H^1(\mathbb{R}^d))} \leq R := C\|u(t_0)\|_{H^1(\mathbb{R}^d)}$ exists at least on the time interval $[t_0, t_0 + T]$, $T > 0$ with

$$T = C^{-1} \|u(t_0)\|_{H^1(\mathbb{R}^d)}^{\frac{q(1-p)}{q-p}},$$

where $q > p$.

Exercise 4

Let $1 + \frac{4}{d} \leq p < 2^* - 1$ and $\kappa = 1$. Let $u_0 \in \Sigma$ and $u \in C(\mathbb{R}; \Sigma)$ be the global-in-time solution of (NLS) given in Theorem 2.12 from the lecture notes. Apply the operator $P = x + 2it\nabla$ to the Duhamel formula

$$u(t,x) = S(t)u_0 - i \int_0^t S(t-s)|u|^{p-1}u(s, \cdot) \, ds, \qquad \text{(Duhamel)}$$

and show

$$\|(x + 2it\nabla)u\|_{L^q([0,\infty); L^{p+1}(\mathbb{R}^d))} < +\infty$$

where $(q, p+1)$ is an admissible exponent pair.