Summer term 2022
Dispersive equations
Exercise sheet 6
Due date: 15.07.2022

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Exercise 1

Let $t \in \mathbb{R}$, $h \in C_0^\infty(\mathbb{R}^d;\mathbb{R})$ a radial, real-valued function and $u \geq 0$ a real-valued function. Show that

$$\int_{\mathbb{R}^d} |u + th|^{p+1}\, dx = \int_{\mathbb{R}^d} u^{p+1}\, dx + (p+1)t \int_{\mathbb{R}^d} hu^p\, dx + o(|t|) \text{ as } t \to 0.$$

Exercise 2

Let $\theta_\varepsilon = e^{\frac{|x|}{1+\varepsilon|x|}}$, $\varepsilon > 0$ be a bounded, Lipschitz continuous function with $|\nabla \theta_\varepsilon|^2 \leq \theta_\varepsilon^2$, a.e.

Let $1 < p < 2^* - 1$ and v satisfies

$$\Delta v - v + v^p = 0, \quad v \geq 0, \quad v \in H_r^1.$$

Show that

$$\int_{\mathbb{R}^d} e^{|x|} v^2\, dx < \infty \tag{1}$$

and

$$\int_{\mathbb{R}^d} e^{|x|} |\nabla v|^2\, dx < \infty. \tag{2}$$

Hint: To show (1) we test the above equation by $\theta_\varepsilon v$, then take $\varepsilon \to 0$.

To show (2) we apply ∂_{x_j} to the above equation and test it by $\theta_\varepsilon \partial_{x_j} v$, then take $\varepsilon \to 0$.

Exercise 3

Let $\rho_n : [0, \infty) \to [0, M]$ be positive monotone functions and there exists a continuous monotone function $\rho(R)$ such that

$$\forall R > 0, \quad \lim_{n\to\infty} \rho_n(R) = \rho(R).$$

Let $m = \lim_{R\to\infty} \rho(R) \leq M$.

Show that there exists a sequence $R_n \to \infty$ such that

$$m = \lim_{n\to\infty} \rho_n(R_n) = \lim_{n\to\infty} \rho_n(\frac{R_n}{2}) = \lim_{R\to\infty} \rho(R).$$

Exercise 4

Recall that, let $(u_n)_{n\geq 1}$ be a bounded sequence in $H^1(\mathbb{R}^d)$ with $\|u_n\|^2_{L^2(\mathbb{R}^d)} = M > 0$. Then, there exists a subsequence which we still denote by u_n such that one of the following three situations holds:

- **Compactness:** There exists a sequence (y_n) in $\mathbb{R}^d$ such that

$$\forall q \in [2, 2^*), \quad u_n(\cdot - y_n) \to u \text{ in } L^q(\mathbb{R}^d) \text{ as } n \to \infty;$$

- **Evanescence:** $\forall q \in (2, 2^*)$, $u_n \to 0$ in $L^q(\mathbb{R}^d)$ as $n \to \infty$;

- **Dichotomy:** There exist two bounded sequences (v_n), (w_n) with compact supports in $H^1(\mathbb{R}^d)$ and $\alpha \in (0, 1)$, such that

$$\begin{aligned}
&\operatorname{Supp} v_n \cap \operatorname{Supp} w_n = \{\}, \quad d(\operatorname{Supp} v_n, \operatorname{Supp} w_n) \to \infty \text{ as } n \to \infty,\\
&\|v_n\|^2_{L^2(\mathbb{R}^d)} \to \alpha M, \quad \|w_n\|^2_{L^2(\mathbb{R}^d)} \to (1-\alpha)M, \text{ as } n \to \infty,\\
&\forall q \in [2, 2^*), \quad \|u_n\|^q_{L^q} - \|v_n\|^q_{L^q} - \|w_n\|^q_{L^q} \to 0, \text{ as } n \to \infty,\\
&\liminf_{n\to\infty} (\|\nabla u_n\|^2_{L^2} - \|\nabla v_n\|^2_{L^2} - \|\nabla w_n\|^2_{L^2}) \geq 0.
\end{aligned}$$

Give three examples of bounded H^1 sequences such that compactness/evanescence/dichotomy hold respectively.