Summer term 2022 **Karlsruhe Institute of Technology**
Dispersive equations JProf. Dr. Xian Liao
Exercise sheet 7 M.Sc. Julia Henninger
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Exercise 1

Let $M > 0$, $1 < p < 1 + \frac{4}{d}$ i.e. $s_c = \frac{d}{2} - \frac{2}{p-1} < 0$. We consider the minimisation problem

$$J_M = \inf\{E(u) \,|\, u \in H^1(\mathbb{R}^d),\ \|u\|^2_{L^2(\mathbb{R}^d)} = M\}, \tag{1}$$

where $E(u) = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla u|^2\,dx - \frac{1}{p+1}\int_{\mathbb{R}^d} |u|^{p+1}\,dx$. Let $u \geq 0$ be a minimiser of (1).

i) Show that there exists $\tilde{\mu} = \tilde{\mu}(M) \in \mathbb{R}$ such that

$$\Delta u + u^p = \tilde{\mu} u, \quad u \geq 0, \quad u \in H^1(\mathbb{R}^d). \tag{2}$$

Hint: We follow the idea in the proof of Proposition 4.3 in the Lecture Notes.

ii) Show that the following equalities hold:

$$\frac{1}{2}\int_{\mathbb{R}^d} |\nabla u|^2\,dx = \frac{a}{p+1}\int_{\mathbb{R}^d} u^{p+1}\,dx,$$
$$\tilde{\mu}\frac{1}{2}\int_{\mathbb{R}^d} |u|^2 = \frac{b}{p+1}\int_{\mathbb{R}^d} u^{p+1}\,dx,$$

where $a = \frac{d(p-1)}{4}$, $\quad b = \frac{d+2}{4} - \frac{(d-2)p}{4}$.

Hint: We test (2) by u and $(\frac{d}{2} + x \cdot \nabla)u$ respectively, then combine the results to get the desired equalities.

Exercise 2

Recall the Cauchy problem of the focusing NonLinear Schrödinger equation

$$\begin{cases} i\partial_t u + \Delta u = -|u|^{p-1}u, \\ u|_{t=0} = u_0(x). \end{cases} \tag{NLS}$$

Recall the "strong" stability notion of the solitary wave solution $e^{it}Q(x)$ in H^1 for $1 < p < 1+\frac{4}{d}$: Let $u_0 \in H^1$ and let $u \in C(\mathbb{R}; H^1)$ be the global solution of (NLS) with the initial data u_0. Then for all $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that for all $u_0 \in H^1$:

$$\|u_0 - Q\|_{H^1} < \delta \text{ implies } \sup_{t\in\mathbb{R}} \|u(t,x) - e^{it}Q(x)\|_{H^1} < \varepsilon.$$

Show that the strong stability property does not hold for (NLS) by the following examples:

- By scaling symmetry, for any $\lambda > 0$, there exists a solution $u_\lambda(t,x) = \lambda^{\frac{2}{p-1}}Q(\lambda x)e^{i\lambda^2 t}$ of (NLS) with the initial data $(u_0)_\lambda(x) = \lambda^{\frac{2}{p-1}}Q(\lambda x)$. Show that

  $$\|(u_0)_\lambda - Q\|_{H^1} \to 0 \text{ as } \lambda \to 1,$$

  while for any $\lambda \neq 1$,

  $$\sup_t \|u_\lambda(t,x) - e^{it}Q(x)\|_{H^1} \geq \|Q\|_{H^1}.$$

- By Galilean invariance, for any $v \in \mathbb{R}^d$, there exists a solution $u_v = e^{i(x\cdot v - |v|^2 t + t)} Q(x - 2vt)$ of (NLS) with the initial data $(u_0)_v = e^{iv\cdot x} Q(x)$. Show that

$$\|(u_0)_v - Q\|_{H^1} \to 0 \text{ as } |v| \to 0,$$

while for $v \neq 0$,

$$\sup_t \|u_v(t,x) - e^{it} Q(x)\|_{H^1} \geq \|Q\|_{H^1}.$$