Notes on

Lecture (0157600) - Fourier analysis and its applications to PDEs

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These are short incomplete notes, only for participants of the course Lecture (0157600) at the Karlsruhe Institute for Technology, Summer Term 2019. Corrections are welcome to be sent to

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The following textbooks/notes are recommended:

- H. Bahouri, J.-Y. Chemin and R. Danchin: Fourier analysis and non-linear partial differential equations. Springer, 2011.
- H. Koch: Lecture notes on PDE and modelling. `http://www.math.uni-bonn.de/ag/ana/SoSe2017/V3B2_SS_17_PDEM/pdem.pdf`

If you have more questions about the lecture, you are welcome to come directly to

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1 Fourier transform

In this section we will introduce the Fourier transform in the whole space setting $\mathbb{R}^d$, $d \geqslant 1$.

1.1 Definition on $L^1(\mathbb{R}^d)$

Definition 1.1. *Let $f \in L^1(\mathbb{R}^d;\mathbb{C})$, $d \geqslant 1$. We define its Fourier transform as a function $\hat{f} \in L^\infty(\mathbb{R}^d;\mathbb{C})$ below*

$$\hat{f}(\xi) := \mathcal{F}(f)(\xi) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)\,\mathrm{d}x\,, \quad \forall \xi \in \mathbb{R}^d\,.$$

Proposition 1.1 (Riemann-Lebesgue)**.** *Let $f \in L^1(\mathbb{R}^d;\mathbb{C})$, then its Fourier transform $\hat{f}$ is continuous and satisfies*

$$\lim_{|\xi|\to\infty} |\hat{f}(\xi)| = 0.$$

Proof. **Exercise.** We make use of the continuity of the function in ξ: $e^{-ix\cdot\xi}$ and the Lebesgue convergence theorem to show the continuity of $\hat{f}(\xi)$. The decay at infinity of $\hat{f}(\xi)$ follows from the density argument and integration by parts. □

By Lemma 1.1, the Fourier transform

$$\mathcal{F} : L^1(\mathbb{R}^d) \mapsto C_b(\mathbb{R}^d) \supset C_0(\mathbb{R}^d)$$

is a continuous map and

$$\|\mathcal{F}(f)\|_{L^\infty(\mathbb{R}^d)} \leqslant \frac{1}{(2\pi)^{\frac{d}{2}}} \|f\|_{L^1(\mathbb{R}^d)}.$$

In the above, $C_0(\mathbb{R}^d)$ is the space of continuous functions which decay to 0 at infinity and $C_b(\mathbb{R}^d)$ is the space of bounded continuous functions endowed with the supremum norm.

We next investigate the application of the Fourier transform on $\tau_a(f)$, $e^{ia\cdot x} f$, $f \circ A$, $f * g$.

Lemma 1.1. *Let $a \in \mathbb{R}^d$, $f, g \in L^1(\mathbb{R}^d)$ and A be a real invertible $d \times d$ matrix. Then*

$$\mathcal{F}(\tau_a(f)) = e^{ia\cdot\xi}\mathcal{F}(f), \quad \tau_a(f) = f(x+a),$$
$$\mathcal{F}(e^{ia\cdot x}f) = \mathcal{F}(f)(\xi - a),$$
$$\mathcal{F}(f \circ A) = |\det A|^{-1}\mathcal{F}(f \circ A^{-T}), \text{ and in particular } \mathcal{F}(f(\lambda\cdot)) = \lambda^{-d}(\mathcal{F}(f))(\lambda^{-1}\cdot), \forall \lambda > 0,$$
$$\mathcal{F}(f * g) = (2\pi)^{\frac{d}{2}}\mathcal{F}(f)\mathcal{F}(g),$$
$$\int_{\mathbb{R}^d} f\mathcal{F}(g)\,\mathrm{d}x = \int_{\mathbb{R}^d} \mathcal{F}(f)g\,\mathrm{d}x\,.$$

Proof. **Exercise.** Here the convolution is defined as

$$(f * g)(x) = \int_{\mathbb{R}^d} f(x-y)g(y)\,\mathrm{d}y = \int_{\mathbb{R}^d} f(y)g(x-y)\,\mathrm{d}y\,,$$

and hence

$$\|f * g\|_{L^1} \leqslant \|f\|_{L^1}\|g\|_{L^1}.$$

□

1.2 Schwartz space $\mathcal{S}(\mathbb{R}^d)$

In this subsection we study the Fourier transform of Schwartz functions, i.e. smooth rapidly decaying functions.

Definition 1.2 (Schwartz space)**.** *The Schwartz space $\mathcal{S}(\mathbb{R}^d)$ is the set of the smooth functions $f \in C^\infty(\mathbb{R}^d)$ satisfying for any $k \in \mathbb{N}$*

$$\|f\|_{k,\mathcal{S}} := \sup_{x\in\mathbb{R}^d,\,|\alpha|\leqslant k}(1+|x|^k)|\partial^\alpha f(x)| < \infty.$$

Remark 1.1. ***Exercise.*** *It is equivalent to say that*

$$\mathcal{S}(\mathbb{R}^d) = \{f \in C^\infty(\mathbb{R}^d)\,|\,\sup_{x\in\mathbb{R}^d}|x^\alpha\partial^\beta f(x)| < \infty, \forall \text{ multiindices } \alpha, \beta\}.$$

Notice that for $f \in \mathcal{S}$, $\|f\|_{k,\mathcal{S}}$ may depend on $k \in \mathbb{N}$.

It is also easy to see that

- $\mathcal{S}(\mathbb{R}^d) \subset C^k(\mathbb{R}^d)$ *for all* $k \in \mathbb{N}$ *and* $\mathcal{S}(\mathbb{R}^d) \subset L^p(\mathbb{R}^d)$ *for all* $p \in [1, \infty]$;
- *If* $f \in \mathcal{S}(\mathbb{R}^d)$*, then* $x^\alpha f$, $\partial^\alpha f \in \mathcal{S}(\mathbb{R}^d)$ *for any multiindex* α;
- *If* $f \in \mathcal{S}(\mathbb{R}^d)$ *and* $g \in C_b^\infty(\mathbb{R}^d)$*, then* $fg \in \mathcal{S}(\mathbb{R}^d)$;
- *If* $f \in \mathcal{S}(\mathbb{R}^d)$ *and* $g \in \mathcal{D}'(\mathbb{R}^d) = (C_0^\infty(\mathbb{R}^d))'$ *with compact support, then* $f * g \in \mathcal{S}(\mathbb{R}^d)$;

- *If $f, g \in \mathcal{S}(\mathbb{R}^d)$, then $f * g \in \mathcal{S}(\mathbb{R}^d)$;*
- *The Gaussian function $e^{-\frac{1}{2}|x|^2} \in \mathcal{S}(\mathbb{R}^d)$.*

We say that $f_n \to f$ in $\mathcal{S}$ if $\|f_n - f\|_{k,\mathcal{S}} \to 0$ for all $k \in \mathbb{N}$. We can then introduce a metric $d(\cdot,\cdot)$ on $\mathcal{S}$:

$$d(f,g) := \sum_{k\in\mathbb{N}} 2^{-k} \frac{\|f-g\|_{k,\mathcal{S}}}{1+\|f-g\|_{k,\mathcal{S}}},$$

such that $d(f_n, f) \to 0$ if and only if $f_n \to f$ in $\mathcal{S}$.

Proposition 1.2. *The space $(\mathcal{S}(\mathbb{R}^d), d(\cdot,\cdot))$ is a complete metric space and the space $\mathcal{D}(\mathbb{R}^d) = C_0^\infty(\mathbb{R}^d)$ of smooth compactly supported functions is dense in it. Hence $\mathcal{S}(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$, $\forall p \in [1,\infty)$.*

Proof. **Exercise.** Observe that for the smooth cutoff function χ, $\|f - \chi(R^{-1}\cdot)f\|_{k,\mathcal{S}} \leqslant C_k R^{-1}\|f\|_{k+1,\mathcal{S}} \to 0$ as $R \to \infty$. □

Since $\mathcal{S}(\mathbb{R}^d) \subset L^1(\mathbb{R}^d)$, we can define the Fourier transform on $\mathcal{S}(\mathbb{R}^d)$:

Theorem 1.1. *The Fourier transform maps continuously from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}(\mathbb{R}^d)$: For any integer $k \in \mathbb{N}$, there exist a constant C and an integer $N \in \mathbb{N}$ such that*

$$\|\mathcal{F}(f)\|_{k,\mathcal{S}} \leqslant C\|f\|_{N,\mathcal{S}}, \quad \forall f \in \mathcal{S}(\mathbb{R}^d).$$

Furthermore, the equalities in Lemma 1.1 together with the following equalities hold true:

$$\mathcal{F}((\frac{1}{i}\partial_x)^\alpha f) = \xi^\alpha \mathcal{F}(f), \quad \mathcal{F}(x^\alpha f) = (i\partial_\xi)^\alpha \mathcal{F}(f), \quad \forall \textit{multiindex } \alpha.$$

Proof. It is straightforward (**Exercise**) to use Lebesgue's theorem and integration by parts to show the above two equalities. Then for any $|\alpha| + |\beta| \leqslant k$, there exist C and N (we can simply take $N = k + d + 1$) such that

$$\begin{aligned}\left|\xi^\alpha \partial_\xi^\beta \mathcal{F}(f)\right| &= \left|\mathcal{F}\big((\frac{1}{i}\partial_x)^\alpha (\frac{1}{i}x)^\beta f\big)\right| \leqslant (2\pi)^{-\frac{d}{2}} \|\partial_x^\alpha (x^\beta f)\|_{L^1} \\ &\leqslant \|(1+|x|)^{-d-1}\|_{L^1} \|(1+|x|)^{d+1} \partial_x^\alpha (x^\beta f)\|_{L^\infty} \leqslant C\|f\|_{N,\mathcal{S}}.\end{aligned}$$

□

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Corollary 1.1. *The Fourier transform maps the Gaussian function* $e^{-\frac{1}{2}|x|^2} \in \mathcal{S}(\mathbb{R}^d)$ *to itself:* $\mathcal{F}(e^{-\frac{1}{2}|x|^2}) = e^{-\frac{1}{2}|\xi|^2}$.

Proof. We first notice that

$$(\partial_{x_j} + x_j)e^{-\frac{1}{2}|x|^2} = 0, \quad j = 1, \cdots, d,$$

and hence by Theorem 1.2,

$$i(\xi_j + \partial_{\xi_j})\mathcal{F}(e^{-\frac{1}{2}|x|^2}) = 0, \quad j = 1, \cdots, d.$$

Let $d = 1$, then the function $\phi(\xi) := \mathcal{F}(e^{-\frac{1}{2}|x|^2})(\xi) \in \mathcal{S}(\mathbb{R})$ satisfies the following first-order differential equation

$$\phi' + \xi\phi = 0, \text{ i.e. } (e^{\frac{1}{2}\xi^2}\phi)' = 0,$$

and hence there exists a constant $C \in \mathbb{R}$ such that

$$\phi(\xi) = Ce^{-\frac{1}{2}\xi^2}.$$

In particular,

$$C = \phi(0) = \mathcal{F}(e^{-\frac{1}{2}x^2})(0) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{1}{2}x^2}\, \mathrm{d}x = 1,$$

and thus

$$\mathcal{F}(e^{-\frac{1}{2}x^2}) = \phi(\xi) = e^{-\frac{1}{2}\xi^2}.$$

For $d \geqslant 2$, we simply notice that

$$\begin{aligned}
\mathcal{F}(e^{-\frac{1}{2}|x|^2}) &= \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} e^{-\frac{1}{2}|x|^2}\, \mathrm{d}x \\
&= \frac{1}{(2\pi)^{\frac{d-1}{2}}} \int_{\mathbb{R}^{d-1}} e^{-ix'\cdot\xi'} e^{-\frac{1}{2}|x'|^2}\, \mathrm{d}x' \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{\mathbb{R}} e^{-ix_d\cdot\xi_d} e^{-\frac{1}{2}(x_d)^2}\, \mathrm{d}x_d \\
&= \frac{1}{(2\pi)^{\frac{d-1}{2}}} \int_{\mathbb{R}^{d-1}} e^{-ix'\cdot\xi'} e^{-\frac{1}{2}|x'|^2}\, \mathrm{d}x' e^{-\frac{1}{2}(\xi_d)^2} = \mathcal{F}(e^{-\frac{1}{2}|x'|^2}) e^{-\frac{1}{2}(\xi_d)^2},
\end{aligned}$$

with $x' = (x_1, \cdots, x_{d-1})$, $\xi' = (\xi_1, \cdots, \xi_{d-1})$. An easy induction argument implies the result for $d \geqslant 2$. □

1.3 Inverse Fourier transform

Definition 1.3. *Let $f \in L^1(\mathbb{R}^d)$. We define*

$$\check{f}(x) := \mathcal{F}^{-1}(f)(x) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{ix\cdot\xi} f(\xi)\,\mathrm{d}\xi\,.$$

Theorem 1.2. *Let $f \in \mathcal{S}(\mathbb{R}^d)$, then $\mathcal{F}^{-1}(f)(x) = \mathcal{F}(f)(-x)$ and*

$$\mathcal{F}^{-1}\mathcal{F}(f)(x) = f(x) = \mathcal{F}\mathcal{F}^{-1}(f)(x).$$

The Fourier transform is an automorphism on $\mathcal{S}(\mathbb{R}^d)$.

Proof. As $\lim_{\varepsilon\to 0} e^{-\frac{\varepsilon^2}{2}|x|^2} = 1$ pointwisely, we calculate straightforward

$$\begin{aligned}
\mathcal{F}^{-1}\mathcal{F}(f)(x) &= (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{ix\cdot\xi}\hat{f}(\xi)\,\mathrm{d}\xi = \lim_{\varepsilon\to 0}(2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\frac{\varepsilon^2}{2}|\xi|^2} e^{ix\cdot\xi}\hat{f}(\xi)\,\mathrm{d}\xi \\
&= \lim_{\varepsilon\to 0}(2\pi)^{-d} \int_{\mathbb{R}^d} e^{-\frac{\varepsilon^2}{2}|\xi|^2} e^{ix\cdot\xi} \int_{\mathbb{R}^d} e^{-iy\cdot\xi} f(y)\,\mathrm{d}y\,\mathrm{d}\xi \\
&= \lim_{\varepsilon\to 0}(2\pi)^{-d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{-\frac{\varepsilon^2}{2}|\xi|^2} e^{-i(y-x)\cdot\xi}\,\mathrm{d}\xi\, f(y)\,\mathrm{d}y \\
&= \lim_{\varepsilon\to 0}(2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \mathcal{F}(e^{-\frac{\varepsilon^2}{2}|\xi|^2})(y-x) f(y)\,\mathrm{d}y\,.
\end{aligned}$$

We recall Lemma 1.1 and Corollary 1.1 to derive that

$$(2\pi)^{-\frac{d}{2}}\mathcal{F}(e^{-\frac{\varepsilon^2}{2}|\xi|^2})(z) = (2\pi)^{-\frac{d}{2}}\varepsilon^{-d} e^{-\frac{1}{2\varepsilon^2}|z|^2},$$

which is a Dirac sequence. Hence $\mathcal{F}^{-1}\mathcal{F}(f)(x) = f(x)$ and $\mathcal{F}\mathcal{F}^{-1} = \mathrm{Id}$ follows simply from $\mathcal{F}^{-1}(f)(x) = \mathcal{F}(f)(-x)$. □

Remark 1.2. ***Exercise.*** *If $f \in \mathcal{S}(\mathbb{R}^d)$, then $e^{-\frac{\varepsilon^2}{2}|x|^2} f \to f$ in $\mathcal{S}(\mathbb{R}^d)$ as $\varepsilon \to 0$. Hence $(2\pi)^{-\frac{d}{2}}\varepsilon^{-d} e^{-\frac{1}{2\varepsilon^2}|x|^2} * f \to f$ in $\mathcal{S}(\mathbb{R}^d)$ as $\varepsilon \to 0$.*

Corollary 1.2. *Let $f, g \in \mathcal{S}(\mathbb{R}^d)$, then*

$$\mathcal{F}(f * g) = (2\pi)^{\frac{d}{2}}\mathcal{F}(f)\mathcal{F}(g), \quad \mathcal{F}(fg) = (2\pi)^{-\frac{d}{2}}\mathcal{F}(f) * \mathcal{F}(g),$$

and

$$\int_{\mathbb{R}^d} f\bar{g}\,\mathrm{d}x = \int_{\mathbb{R}^d} \hat{f}\bar{\hat{g}}\,\mathrm{d}\xi\,.$$

In particular, $\|f\|_{L^2} = \|\hat{f}\|_{L^2}$.

Proof. **Exercise**. We show the result by the following equalities from Lemma 1.1:

$$\mathcal{F}(f * g) = (2\pi)^{\frac{d}{2}}\mathcal{F}(f)\mathcal{F}(g), \quad \int_{\mathbb{R}^d} f\hat{g} = \int_{\mathbb{R}^d} \hat{f}g.$$

□

1.4 Tempered distribution space

Definition 1.4. *A tempered distribution on $\mathbb{R}^d$ is a continuous linear map from $\mathcal{S}(\mathbb{R}^d)$ to $\mathbb{C}$. We denote the set of tempered distributions by $\mathcal{S}'(\mathbb{R}^d)$. We say that $T_j \to T$ in $\mathcal{S}'(\mathbb{R}^d)$ if*

$$T_j(f) \to T(f) \quad \text{for any } f \in \mathcal{S}(\mathbb{R}^d).$$

Remark 1.3.
- *By the definition above, a tempered distribution T is a distribution $T \in \mathcal{D}'(\mathbb{R}^d) = (C_0^\infty(\mathbb{R}^d))'$ such that there exists $k \in \mathbb{N}$ and $C \in \mathbb{R}$ s.t.*

$$|T(f)| \leqslant C\|f\|_{k,\mathcal{S}}, \quad \forall f \in \mathcal{D}(\mathbb{R}^d).$$

  Indeed, if $T \in \mathcal{D}'(\mathbb{R}^d)$ such that the above inequality holds, then as $\mathcal{D}(\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}^d)$ is dense, there exists a unique continuation of T as a continuous linear map from $\mathcal{S}(\mathbb{R}^d)$.

- *The convergence $T_j \to T$ in $\mathcal{S}'(\mathbb{R}^d)$ means indeed that there exists $k \in \mathbb{N}$ such that*

$$\sup_{\|f\|_{k,\mathcal{S}} \leqslant 1} |T_j(f) - T(f)| \to 0$$

Definition 1.5. *Let $T \in \mathcal{D}'(\mathbb{R}^d)$. We define the support of T:* Supp (T) *as the complement of the following set*

$$\{x \in \mathbb{R}^d \mid \exists r > 0 \text{ s.t. } T(f) = 0, \quad \forall f \in C_0^\infty(B(0,r))\}.$$

Example 1.1.
- *Every function $g \in L^1(\mathbb{R}^d)$ defines a tempered distribution $T_g \in \mathcal{S}'(\mathbb{R}^d)$ as*

$$T_g(f) = \int_{\mathbb{R}^d} gf \,\mathrm{d}x\,, \quad \forall f \in \mathcal{S}(\mathbb{R}^d).$$

  Similarly it holds for $g \in L^p$ for any $p \in [1, \infty]$.

- *Let us denote the set of distributions with compact support by $\mathcal{E}'(\mathbb{R}^d)$, which is the dual space of $C^\infty(\mathbb{R}^d)$ endowed with the seminorms $\sup_{|\alpha| \leqslant k} \|\partial^\alpha f\|_{L^\infty(B(0,k))}$, $k \in \mathbb{N}$. Then $\mathcal{E}'(\mathbb{R}^d) \subset \mathcal{S}'(\mathbb{R}^d)$ and in particular the Dirac function $\delta_0 \in \mathcal{S}'(\mathbb{R}^d)$.*

If $A : \mathcal{S}(\mathbb{R}^d) \mapsto \mathcal{S}(\mathbb{R}^d)$ is a linear operator, then by duality we can define the operator $A^t : \mathcal{S}'(\mathbb{R}^d) \mapsto \mathcal{S}'(\mathbb{R}^d)$ as follows:

$$(A^t T)(f) = T(Af), \quad \forall T \in \mathcal{S}'(\mathbb{R}^d), \quad f \in \mathcal{S}(\mathbb{R}^d).$$

Definition 1.6. *Let $\phi \in C^\infty$ with at most polynomial growth, $f \in \mathcal{S}(\mathbb{R}^d)$ and $T \in \mathcal{S}'(\mathbb{R}^d)$. Then we define the following operators on $\mathcal{S}'(\mathbb{R}^d)$:*

- *The product by ϕ : $(\phi T)(f) = T(\phi f)$,*
- *The derivative ∂_{x_j} : $(\partial_{x_j} T)(f) = -T(\partial_{x_j} f)$,*
- *The convolution with f : $(T * f)(x) = T(f(x - \cdot))$,*
- *The Fourier transform $\mathcal{F}$: $\mathcal{F}(T)(f) = T(\mathcal{F}(f))$,*
- *The inverse Fourier transform $\mathcal{F}^{-1}(T)(f) = T(\mathcal{F}^{-1}(f))$.*

Remark 1.4. *We can simply check (**Exercise**) the above equalities directly when $T = T_g$, $g \in \mathcal{S}(\mathbb{R}^d)$.*

Hence by Theorem 1.2 the Fourier transform is an automorphism on $\mathcal{S}'(\mathbb{R}^d)$.

[29.04.2019]

[06.05.2019]

Proposition 1.3. *$\mathcal{S}(\mathbb{R}^d)$ is dense in $\mathcal{S}'(\mathbb{R}^d)$.*

Proof. By Remark 1.2,

$$(2\pi)^{-\frac{d}{2}}\varepsilon^{-d}e^{-\frac{1}{2\varepsilon^2}|\cdot|^2} * (e^{-\frac{\delta^2}{2}|x|^2} f) \to f \text{ in } \mathcal{S}(\mathbb{R}^d), \text{ as } \varepsilon \to 0, \quad \delta \to 0.$$

By the above definition, for any $T \in \mathcal{S}'(\mathbb{R}^d)$, there exist k and C such that

$$\begin{aligned}\big|\big(T * g\big)(x)\big| = \big|T(g(x - \cdot))\big| &\leqslant C\|g(x - \cdot)\|_{k,\mathcal{S}} \\ &= C\sup{}_{y\in\mathbb{R}^d, |\alpha|\leqslant k}(1 + |y|^k)|\partial_y^\alpha g(x - y)|, \quad \forall g \in \mathcal{S}(\mathbb{R}^d),\end{aligned}$$

which is a smooth function with at most polynomial growth, and for any $f, g \in \mathcal{S}$,

$$(T * g)(f) = \int_{\mathbb{R}^d} T(g(x - \cdot))\, f(x)\,\mathrm{d}x \; = T\Big(\int_{\mathbb{R}^d} g(x - \cdot)f(x)\,\mathrm{d}x\Big) = T(g(-\cdot) * f).$$

Hence for any $T \in \mathcal{S}'$,

$$\begin{aligned}&T\Big((2\pi)^{-\frac{d}{2}}\varepsilon^{-d}e^{-\frac{1}{2\varepsilon^2}|\cdot|^2} * (e^{-\frac{\delta^2}{2}|x|^2} f)\Big) \\ &= \Big(T * (2\pi)^{-\frac{d}{2}}\varepsilon^{-d}e^{-\frac{1}{2\varepsilon^2}|\cdot|^2}\Big)(e^{-\frac{\delta^2}{2}|x|^2} f) \\ &= \Big(e^{-\frac{\delta^2}{2}|x|^2}\big((2\pi)^{-\frac{d}{2}}\varepsilon^{-d}e^{-\frac{1}{2\varepsilon^2}|\cdot|^2} * T\big)\Big)(f) \to T(f), \text{ as } \varepsilon \to 0, \quad \delta \to 0.\end{aligned}$$

Thus

$$\mathcal{S} \ni e^{-\frac{\delta^2}{2}|x|^2}\big((2\pi)^{-\frac{d}{2}}\varepsilon^{-d}e^{-\frac{1}{2\varepsilon^2}|\cdot|^2} * T\big) \to T \text{ in } \mathcal{S}'.$$

□

Theorem 1.3. *The Fourier transform defines a unitary operator from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$.*

Proof. Let $f \in L^2(\mathbb{R}^d)$ and let $(f_n)_n \subset \mathcal{S}(\mathbb{R}^d)$ converge to f in $L^2(\mathbb{R}^d)$. Then by Corollary 1.2, $\|\hat{f_m} - \hat{f_n}\|_{L^2} = \|f_m - f_n\|_{L^2}$ and hence $(\hat{f}_n)_n \subset \mathcal{S}(\mathbb{R}^d)$ is a Cauchy sequence in L^2. We define the unique limit of $(\hat{f}_n)_n$ in L^2 as the Fourier transform of f. This is a unitary operator from $\int_{\mathbb{R}^d} f\bar{g} = \int_{\mathbb{R}^d} \hat{f}\bar{\hat{g}}$. □

Proposition 1.4. *Let $T \in \mathcal{S}'(\mathbb{R}^d)$. Then* $\mathrm{Supp}\,(T) = \{0\} \Leftrightarrow \mathcal{F}(T)$ *is a polynomial.*

Proof. "$\Leftarrow$" is straightforward (**Exercise**).

Now let $\mathrm{Supp}\,(T) = \{0\}$. For any $\varepsilon \in (0,1)$, we take a cutoff function $\chi_\varepsilon = \chi(\varepsilon^{-1}\cdot) \in C_0^\infty(B(0,\varepsilon))$. Then $T((1-\chi_\varepsilon)f) = 0$ for any $f \in \mathcal{S}(\mathbb{R}^d)$ and hence $T(f) = T(\chi_\varepsilon f)$. As $T \in \mathcal{S}'(\mathbb{R}^d)$, there exist k, C such that

$$|T(f)| = |T(\chi_\varepsilon f)| \leqslant C\|\chi_\varepsilon f\|_{k,\mathcal{S}}.$$

Let $f \in \mathcal{S}(\mathbb{R}^d)$ such that $\partial^\alpha f(0) = 0$ for any $|\alpha| \leqslant k$, then for any $|x| \leqslant \varepsilon < 1$, by Taylor's expansion formula,

$$\begin{aligned}
&|f(x)| \leqslant C_0 \sup_{|\beta|=k+1} \|\partial^\beta f\|_{L^\infty} |x|^{k+1},\\
&|\partial^\gamma f(x)| \leqslant C_0 \sup_{|\beta|=k+1} \|\partial^\beta f\|_{L^\infty} |x|^{k+1-|\gamma|}, \quad \forall |\gamma| \leqslant k.
\end{aligned}$$

Hence

$$|T(f)| \leqslant C\|\chi(\varepsilon^{-1}\cdot)f\|_{k,\mathcal{S}} \leqslant C_1\varepsilon\|f\|_{k+1,\mathcal{S}},$$

which tends to 0 as $\varepsilon \to 0$. Thus $T(f) = 0$.

Let $f \in \mathcal{S}(\mathbb{R}^d)$. Then by $\mathrm{Supp}\,(T) = \{0\}$ and Taylor's formula again,

$$T(f) = T\big(f - \sum_{|\alpha|\leqslant k} \frac{1}{\alpha!}\partial^\alpha f(0)x^\alpha\big) + \sum_{|\alpha|\leqslant k} \frac{1}{\alpha!}\partial^\alpha f(0) T(\chi x^\alpha),$$

where the first summand on the righthand side vanishes by the above argument. Let $g = \mathcal{F}^{-1}f$, then

$$\begin{aligned}
\hat{T}(g) = T(\hat{g}) = \sum_{|\alpha|\leqslant k} \frac{1}{\alpha!}(\partial^\alpha \hat{g})(0) T(\chi x^\alpha) &= \sum_{|\alpha|\leqslant k} \frac{1}{\alpha!}\big(T(\chi x^\alpha)\big)\int_{\mathbb{R}^d} (-ix)^\alpha g(x)\,\mathrm{d}x\\
&= \Big(\sum_{|\alpha|\leqslant k} \frac{1}{\alpha!}\big(T(\chi x^\alpha)\big)(-ix)^\alpha\Big)(g).
\end{aligned}$$

□

1.5 Examples

Example 1.2. *Calculate the Fourier transform of the delta function* $\hat{\delta} = (2\pi)^{-\frac{d}{2}}$*, and hence* $\widehat{\partial^\alpha \delta} = (2\pi)^{-\frac{d}{2}}(i\xi)^\alpha$*. Thus by Proposition 1.4, the tempered distribution* T *with* $\mathrm{Supp}\,(T) = \{0\}$ *can only be a linear combination of* δ *and its derivatives:* $T = \sum_{|\alpha|\leqslant k} a_\alpha \partial^\alpha \delta_0$.

Indeed, it is easy to calculate

$$\hat{\delta}(f) = \delta(\hat{f}) = \hat{f}(0) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \hat{f}(\xi)\,\mathrm{d}\xi \; = \big((2\pi)^{-\frac{d}{2}}\big)(f).$$

Example 1.3. *When* $d = 1$*, then* $\mathcal{F}(e^{-|x|}) = \sqrt{\frac{2}{\pi}} \frac{1}{1+\xi^2}$ *(**Exercise**).*

Example 1.4. *When* $\sigma \in (0, d)$*, then* $\mathcal{F}(|x|^{-\sigma}) = c|\xi|^{\sigma-d}$ *for some constant* c*. (**Exercise**)*

Let $d \geqslant 2$. Define the operators

$$R = \sum_{j=1}^d x_j \partial_{x_j}, \quad Z_{j,k} = x_j \partial_{x_k} - x_k \partial_{x_j},$$

such that

$$|x|^2 \partial_{x_k} = x_k R + \sum_{j=1}^d x_j Z_{j,k},$$

and

$$R(|x|^{-\sigma}) = -\sigma |x|^{-\sigma}, \quad Z_{j,k}(|x|^{-\sigma}) = 0.$$

Then we show that

$$\begin{aligned}
&R(\mathcal{F}(|x|^{-\sigma})) = -\mathcal{F}(R(|x|^{-\sigma})) - d\mathcal{F}(|x|^{-\sigma}) = (\sigma - d)\mathcal{F}(|x|^{-\sigma}),\\
&Z_{j,k}(\mathcal{F}(|x|^{-\sigma})) = 0,
\end{aligned}$$

such that

$$|\xi|^2 \partial_{x_k}(|\xi|^{d-\sigma}\mathcal{F}(|x|^{-\sigma})) = \xi_k R(|\xi|^{d-\sigma}\mathcal{F}(|x|^{-\sigma})) + \sum_{j=1}^d \xi_j Z_{j,k}(|\xi|^{d-\sigma}\mathcal{F}(|x|^{-\sigma})) = 0.$$

Therefore the distribution $\nabla(|\xi|^{d-\sigma}\mathcal{F}(|x|^{-\sigma}))$ is supported at $\{0\}$ such that $T := |\xi|^{d-\sigma}\mathcal{F}(|x|^{-\sigma}) - c$ is supported at $\{0\}$ for some constant c. Thus by Example 1.2, $T = \sum_{|\alpha|\leqslant k} a_\alpha \partial^\alpha \delta_0$ and

$$0 = RT = \sum_{|\alpha|\leqslant k} a_\alpha R\big(\partial^\alpha \delta_0\big) = -\sum_{|\alpha|\leqslant k} a_\alpha (d + |\alpha|)\partial^\alpha \delta_0,$$

where we used $R(\partial^\alpha\delta_0) = -(d+|\alpha|)\partial^\alpha\delta_0$ (from the fact $\partial^\alpha\delta_0 = (\widehat{\frac{1}{i}x)^\alpha(2\pi})^{-\frac{d}{2}})$. Hence $a_\alpha = 0$, $T = 0$ and $\mathcal{F}(|x|^{-\sigma}) = c|\xi|^{\sigma-d}$.

For $d = 1$, we simply make use of the operator $R = x\frac{d}{dx}$.

Remark 1.5. *If f is homogeneous of degree m: $f(\lambda x) = \lambda^m f(x)$, then $\hat{f}$ is homogeneous of degree $-(m+d)$: $\hat{f}(\lambda^{-1}\xi) = \lambda^{m+d}\hat{f}(\xi)$. This coincides with Example 1.4.*

1.6 Functions with compactly supported Fourier transforms

We have already seen from Proposition 1.4 that if a tempered distribution's Fourier transform is compacted supported at $\{0\}$, then it is a polynomial. We now investigate the relations between the L^p functions and their derivatives when their Fourier transforms are compactly supported on a ball or on an annulus.

Lemma 1.2 (Bernstein)**.** *Let $\mathcal{B} = \{\xi \in \mathbb{R}^d \mid |\xi| \leqslant R\}$ be a ball centered at 0 with radius $R > 0$ and $\mathcal{C} = \{\xi \in \mathbb{R}^d \mid 0 < r_1 \leqslant |\xi| \leqslant r_2\}$ be an annulus. Then there exists a constant C such that the following facts hold for any $k \in \mathbb{N}$, $\lambda > 0$, $p, q \in [1, \infty]$ with $p \leqslant q$ and $u \in L^p(\mathbb{R}^d)$:*

$$\text{Supp}\,(\hat{u}) \subset \lambda\mathcal{B} \Rightarrow \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^q} \leqslant C^{k+1}\lambda^{k+\frac{d}{p}-\frac{d}{q}}\|u\|_{L^p},$$
$$\text{Supp}\,(\hat{u}) \subset \lambda\mathcal{C} \Rightarrow C^{-k-1}\lambda^k\|u\|_{L^p} \leqslant \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^p} \leqslant C^{k+1}\lambda^k\|u\|_{L^p}.$$

Proof. By scaling argument (**Exercise**) we can restrict ourselves to the case with $\lambda = 1$ and by density argument it remains to consider $u \in \mathcal{S}(\mathbb{R}^d)$.

Let $\chi \in \mathcal{D}(\mathbb{R}^d)$ be a cutoff function with value 1 on $\mathcal{B}$. If $\text{Supp}\,(\hat{u}) \subset \mathcal{B}$, then

$$\hat{u} = \chi\hat{u} \Rightarrow u = (2\pi)^{-\frac{d}{2}}(\check{\chi} * u) \Rightarrow \partial^\alpha u = (2\pi)^{-\frac{d}{2}}(\partial^\alpha\check{\chi}) * u,$$

and hence by Young's inequality,

$$\|\partial^\alpha u\|_{L^q} \leqslant (2\pi)^{-\frac{d}{2}}\|\partial^\alpha\check{\chi}\|_{L^r}\|u\|_{L^p}, \quad 1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{r}.$$

Obviously $\chi \in \mathcal{S}$ and $\partial^\alpha\check{\chi} \in \mathcal{S} \subset L^r$, while we can also estimate

$$\|f\|_{L^r} \leqslant \|f\|_{L^1} + \|f\|_{L^\infty} \leqslant C\|(1+|x|^2)^d f\|_{L^\infty} + \|f\|_{L^\infty} \leqslant C\|(1-\Delta)^d\hat{f}\|_{L^1},$$

we derive $\|\partial^\alpha\check{\chi}\|_{L^r} \leqslant C\|(1-\Delta)^d(\xi^\alpha\chi)\|_{L^1} \leqslant C^{k+1}$, such that $\|\partial^\alpha u\|_{L^q} \leqslant C^{k+1}\|u\|_{L^p}$ follows.

[06.05.2019]

[07.05.2019]

If $\mathrm{Supp}\,(\hat{u}) \subset \mathcal{C}$, then we can take $\varphi \in \mathcal{D}(\mathbb{R}^d \backslash \{0\})$ with $\varphi = 1$ on $\mathcal{C}$ such that

$$\hat{u} = \varphi \hat{u} = \varphi |\xi|^{-2k} \sum_{|\alpha|=k} (-i\xi)^\alpha (i\xi)^\alpha \hat{u} = \sum_{|\alpha|=k} \big(\varphi \frac{(-i\xi)^\alpha}{|\xi|^{2k}}\big) \widehat{\partial^\alpha u}.$$

Then by a similar argument as above we have

$$\|u\|_{L^p} \leqslant \sum_{|\alpha|=k} \Big\| \mathcal{F}^{-1}\big(\varphi \frac{(-i\xi)^\alpha}{|\xi|^{2k}}\big) * \partial^\alpha u \Big\|_{L^p} \leqslant \sum_{|\alpha|=k} \Big\| \mathcal{F}^{-1}\big(\varphi \frac{(-i\xi)^\alpha}{|\xi|^{2k}}\big) \Big\|_{L^1} \|\partial^\alpha u\|_{L^p},$$

where

$$\begin{aligned} \big\| \mathcal{F}^{-1}\big(\varphi \frac{(-i\xi)^\alpha}{|\xi|^{2k}}\big) \big\|_{L^1} &\leqslant C \big\| (1+|x|^2)^d \mathcal{F}^{-1}\big(\varphi \frac{(-i\xi)^\alpha}{|\xi|^{2k}}\big) \big\|_{L^\infty} \\ &\leqslant C \big\| (1-\Delta)^d \big(\varphi \frac{(-i\xi)^\alpha}{|\xi|^{2k}}\big) \|_{L^1} \leqslant C^{k+1}. \end{aligned}$$

□

We also have the following lemma describing the action of the heat semi-group on L^p functions whose Fourier transform are supported on an annulus.

Lemma 1.3. *Let $\mathcal{C}$ be an annulus as in Lemma 1.2. Then there exists a constant C such that the following fact holds true for any $t > 0$, $\lambda > 0$, $p \in [1, \infty]$ and $u \in L^p$:*

$$\mathrm{Supp}\,(\hat{u}) \subset \lambda \mathcal{C} \Rightarrow \|e^{t\Delta} u\|_{L^q} \leqslant C e^{-C^{-1} t \lambda^2} \lambda^{\frac{d}{p} - \frac{d}{q}} \|u\|_{L^p}, \quad \forall q \in [p, \infty].$$

Proof. By scaling argument (**Exercise**) we can restrict ourselves to the case $\lambda = 1$. Take $\varphi \in \mathcal{D}(\mathbb{R}^d \backslash \{0\})$ as in the above proof, then if $\mathrm{Supp}\,(\hat{u}) \subset \mathcal{C}$,

$$\hat{u} = \varphi \hat{u} \Rightarrow \widehat{e^{t\Delta} u} = \varphi e^{-t|\xi|^2} \hat{u}.$$

It remains to show (**Exercise**)

$$\|\mathcal{F}^{-1}\big(\varphi e^{-t|\xi|^2}\big)\|_{L^r} \leqslant C \|(1-\Delta)^d (\varphi e^{-t|\xi|^2})\|_{L^1} \leqslant C e^{-C^{-1} t}.$$

□

It is straightforward (**Exercise**) to derive from Lemma 1.2 and Lemma 1.3 that

Corollary 1.3. *Let $\mathcal{C}$ be an annulus as above. Then there exists a constant C such that for any $T > 0$, $\lambda > 0$, $1 \leqslant p \leqslant q \leqslant \infty$, $1 \leqslant b \leqslant a \leqslant \infty$ there hold*

$$\mathrm{Supp}\,(\hat{u_0}) \subset \lambda\mathcal{C} \underset{u|_{t=0}=u_0}{\overset{(\partial_t-\mu\Delta)u=0}{\Longrightarrow}} \|u\|_{L^a([0,T];L^q)} \leqslant C(\mu\lambda^2)^{-\frac{1}{a}}\lambda^{\frac{d}{p}-\frac{d}{q}}\|u_0\|_{L^p},$$

and

$$\mathrm{Supp}\,(\hat{f}(t,\cdot)) \subset \lambda\mathcal{C},\ \forall t \in [0,T]$$
$$\underset{u|_{t=0}=0}{\overset{(\partial_t-\mu\Delta)u=f}{\Longrightarrow}} \|u\|_{L^a([0,T];L^q)} \leqslant C(\mu\lambda^2)^{-1+\frac{1}{b}-\frac{1}{a}}\lambda^{\frac{d}{p}-\frac{d}{q}}\|f\|_{L^b([0,T];L^p)}.$$

2 Littlewood-Paley theory

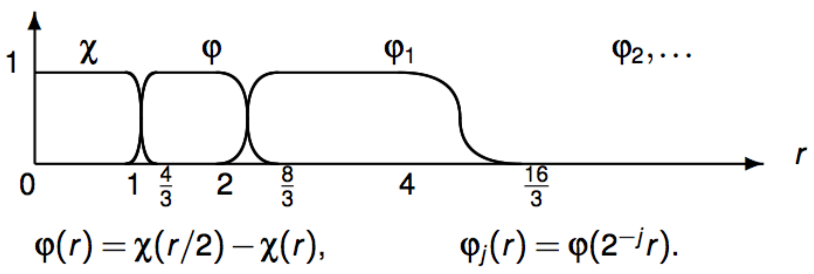
We have seen from Berstein's inequalities in Lemma 1.2 that if the Fourier transform $\hat{u}$ of a function $u \in L^p(\mathbb{R}^d)$ is compactly supported on an annulus, then the application of the derivatives ∇ on u works as a multiplication of λ on u: $\|\nabla u\|_{L^p} \sim \lambda\|u\|_{L^p}$, with λ denoting the size of $\mathrm{Supp}\,(\hat{u})$. We hence introduce the following dyadic partition of unity

$$1 = \chi(\xi) + \sum_{j\geqslant 0}\varphi_j(\xi), \quad \varphi_j(\xi) = \varphi(2^{-j}\xi) \quad \forall \xi \in \mathbb{R}^d, \tag{2.1}$$

where the radial functions

$$\chi \in \mathcal{D}(\mathcal{B}),\ \mathcal{B} = \{\xi \in \mathbb{R}^d \mid |\xi| \leqslant \frac{4}{3}\},$$
$$\varphi = \chi(\cdot/2) - \chi \in \mathcal{D}(\mathcal{C}),\ \mathcal{C} = \{\xi \in \mathbb{R}^d \mid 1 \leqslant |\xi| \leqslant \frac{8}{3}\},$$

take the values in the interval $[0,1]$ as follows:



We can then do the Littlewood-Paley decomposition (formally) for $u \in L^p(\mathbb{R}^d)$ as follows:

$$u = \Delta_{-1}u + \sum_{j\geqslant 0}\Delta_j u, \text{ with } \widehat{\Delta_{-1}u} = \chi(\xi)\hat{u}(\xi), \quad \widehat{\Delta_j u} = \varphi_j(\xi)\hat{u}(\xi), \tag{2.2}$$

such that by Bernstein's inequalities

$$\|\Delta_j u\|_{L^q} \leqslant C 2^{j(\frac{d}{p}-\frac{d}{q})} \|\Delta_j u\|_{L^p}, \quad \forall j \geqslant -1, \quad \forall q \geqslant p,$$
$$\sup_{|\alpha|=k} \|\partial^\alpha \Delta_j u\|_{L^p} \geqslant C^{-1} 2^{jk} \|\Delta_j u\|_{L^p}, \quad \forall j \geqslant 0.$$

We also introduce the low-frequency cut-off operator S_j as follows:

$$S_j u = \sum_{j' \leqslant j-1} \Delta_{j'} u, \text{ i.e. } \widehat{S_j u} = \chi(2^{-j} \cdot) \hat{u}, \quad j \geqslant 0, \tag{2.3}$$

and we have

$$\|S_j u\|_{L^q} \leqslant C 2^{j(\frac{d}{p}-\frac{d}{q})} \|u\|_{L^p}, \quad \forall j \geqslant 0, \quad \forall q \geqslant p,$$

and $S_j \to \mathrm{Id}$, $j \to \infty$ on $\mathcal{S}'(\mathbb{R}^d)$:

Proposition 2.1. *Let $u \in \mathcal{S}'(\mathbb{R}^d)$, then*

$$S_j u \to u \text{ in } \mathcal{S}'(\mathbb{R}^d), \text{ as } j \to \infty.$$

Proof. **Exercise.** By duality and Theorem 1.2 it suffices to show

$$\widehat{S_j f} = \chi(2^{-j} \cdot) \hat{f} \to \hat{f} \text{ in } \mathcal{S}(\mathbb{R}^d).$$

□

Hence the Littlewood-Paley decomposition (2.2): Id $= \sum_{j \geqslant -1} \Delta_j$ is well-defined on $\mathcal{S}'(\mathbb{R}^d)$.

2.1 Homogeneous Besov spaces

It is easy to notice from the dyadic partition of unity (2.1) that

$$1 = \sum_{j \in \mathbb{Z}} \varphi_j(\xi), \quad \forall \xi \neq 0.$$

We now restrict ourselves to the following tempered distributions whose Fourier transforms vanish at the origin in the following sense:

Definition 2.1. *We denote by $\mathcal{S}'_h(\mathbb{R}^d)$ the space of tempered distributions $u \in \mathcal{S}(\mathbb{R}^d)$ such that*

$$\lim_{\lambda \to \infty} \|\theta(\lambda D) u\|_{L^\infty} = 0 \text{ with } \widehat{\theta(\lambda D)} u = \theta(\lambda \xi) \hat{u}(\xi), \quad \forall \theta \in \mathcal{D}(\mathbb{R}^d).$$

Remark 2.1. *Any tempered distribution* $u \in \mathcal{S}'$ *with* $\hat{u} \in L^1_{\text{loc}}$ *belongs to* $\mathcal{S}'_h$. *Any* L^p, $p \in [1, \infty)$ *function* u *belongs to* $\mathcal{S}'_h$ *since* $\|\theta(\lambda D)u\|_{L^\infty} = \|\lambda^{-d}\check{\theta}(\lambda^{-1}\cdot) * u\|_{L^\infty} \leqslant \|u\|_{L^p}\|\lambda^{-d}\check{\theta}(\lambda^{-1}\cdot)\|_{L^{p'}} \to 0$ *as* $\lambda \to \infty$, *as long as* $p' \neq 1$.

The polynomial $P \neq 0$ *does not belong to* $\mathcal{S}'_h$ *as* $\theta(\lambda\cdot)\hat{P} = \theta(0)\hat{P}$.

We then denote by $\dot{\Delta}_j$, $\dot{S}_j$, $j \in \mathbb{Z}$ the following operators (to distinguish from (2.2) and (2.3) where $j \in \mathbb{N}$):

$$\begin{aligned} \dot{\Delta}_j u &= \varphi(2^{-j}D)u \text{ with } \widehat{\varphi(2^{-j}D)u} = \varphi(2^{-j}\xi)\hat{u}(\xi), \quad j \in \mathbb{Z}, \\ \dot{S}_j u &= \chi(2^{-j}D)u \text{ with } \widehat{\chi(2^{-j}D)u} = \chi(2^{-j}\xi)\hat{u}(\xi), \quad j \in \mathbb{Z}, \end{aligned} \tag{2.4}$$

such that

$$u = \sum_{j \in \mathbb{Z}} \dot{\Delta}_j u \text{ and } \lim_{j \to -\infty} \dot{S}_j u = 0 \text{ in } \mathcal{S}', \text{ if } u \in \mathcal{S}'_h. \tag{2.5}$$

Definition 2.2 (Homogeneous Besov spaces)**.** *Let* $s \in \mathbb{R}$, $(p, r) \in [1, \infty]^2$. *We denote by* $\dot{B}^s_{p,r}(\mathbb{R}^d)$ *the set of tempered distributions* $u \in \mathcal{S}'_h$ *such that*

$$\|u\|_{\dot{B}^s_{p,r}} = \left\| \left(2^{js}\|\dot{\Delta}_j u\|_{L^p(\mathbb{R}^d)} \right)_{j \in \mathbb{Z}} \right\|_{\ell^r} < \infty. \tag{2.6}$$

Example 2.1. *The function* $|x|^{-\sigma}$, $\sigma \in (0, d)$ *belongs to* $\dot{B}^{\frac{d}{p}-\sigma}_{p,\infty}$ *for all* $p \in [1, \infty]$, *but not to* $\dot{B}^{\frac{d}{p}-\sigma}_{p,r}$ *for any* $r \in [1, \infty)$.

Let us take $p = 1$ and show $|x|^{-\sigma} \in \dot{B}^{d-\sigma}_{1,\infty}$ for $\sigma \in (0, d)$ (**Exercise**). Indeed, we take a smooth cutoff function ρ to decompose the function $f = |x|^{-\sigma}$ into

$$f = f_1 + f_2, \quad f_1 = \rho f \in L^1, \quad f_2 = (1 - \rho)f \in L^q, \quad \forall q > \frac{d}{\sigma},$$

such that $f \in \mathcal{S}'_h$ by Remark 2.1. Let $h = \mathcal{F}^{-1}\varphi$, then we calculate

$$\dot{\Delta}_j f = (2\pi)^{-\frac{d}{2}} 2^{jd} h(2^j\cdot) * f = 2^{j\sigma}(\dot{\Delta}_0 f)(2^j\cdot),$$

such that $2^{j(d-\sigma)}\|\dot{\Delta}_j f\|_{L^1} = \|\dot{\Delta}_0 f\|_{L^1}$. It remains to show $\dot{\Delta}_0 f \in L^1$ by Bernstein's inequality:

$$\begin{aligned} &\|\dot{\Delta}_0 f_1\|_{L^1} = \|(2\pi)^{-\frac{d}{2}} h * f_1\|_{L^1} \leqslant \|h\|_{L^1}\|f_1\|_{L^1} \leqslant C\|f_1\|_{L^1} < \infty, \\ &\|\dot{\Delta}_0 f_2\|_{L^1} \leqslant C_k\|\dot{\Delta}_0(D^k f_2)\|_{L^1} \leqslant C_k\|D^k((1 - \rho)f)\|_{L^1} < \infty \text{ if } k > d - \sigma. \end{aligned}$$

However, as $\|\dot\Delta_0 f\|_{L^1} \neq 0$ (otherwise $f = 0$), $(2^{j(d-\sigma)}\|\dot\Delta_j f\|_{L^1}))_j \notin \ell^r$ for any $r < \infty$. By Proposition 2.2 (see below), $f \in \dot B^{d-\sigma}_{1,\infty} \subset \dot B^{\frac{d}{p}-\sigma}_{p,\infty}$ for any $p \in [1, \infty]$. On the other hand, as

$$2^{j(\frac{d}{p}-\sigma)}\|\dot\Delta_j f\|_{L^p} = 2^{j\frac{d}{p}}\|(\dot\Delta_0 f)(2^j\cdot)\|_{L^p} = \|\dot\Delta_0 f\|_{L^p},$$

$(2^{j(\frac{d}{p}-\sigma)}\|\dot\Delta_j f\|_{L^p}))_j \notin \ell^r$ for any $r < \infty$.

[07.05.2019]

[13.05.2019]

Theorem 2.1. *The homogeneous Besov space $\dot B^s_{p,r} \subset \mathcal{S}'_h$, $s \in \mathbb{R}$, $(p, r) \in [1, \infty]^2$ is a normed space. If $(p, r) \in [1, \infty)^2$, then the space $\mathcal{S}_0(\mathbb{R}^d) := \{f \in \mathcal{S}(\mathbb{R}^d) \,|\, \mathrm{Supp}\,(\hat f) \cap \{0\} = \varnothing\}$ is dense in $\dot B^s_{p,r}(\mathbb{R}^d)$.*

The definition of the homogeneous Besov space $\dot B^s_{p,r}$ is independent of the choice of the function φ in the dyadic partition. Let $\mathcal{C}'$ be an annulus and $(u_j)_{j\in\mathbb{Z}}$ be a sequence of functions such that

$$\mathrm{Supp}\,(\hat{u_j}) \subset 2^j\mathcal{C}', \quad \big(2^{js}\|u_j\|_{L^p}\big)_{j\in\mathbb{Z}} \in \ell^r(\mathbb{Z}), \quad \sum_{j\in\mathbb{Z}} u_j \to u \text{ in } \mathcal{S}' \text{ with } u \in \mathcal{S}'_h,$$

then $u \in \dot B^s_{p,r}$ and

$$\|u\|_{\dot B^s_{p,r}} \leqslant C_s\Big\|\big(2^{js}\|u_j\|_{L^p}\big)_{j\in\mathbb{Z}}\Big\|_{\ell^r}.$$

If furthermore $s > 0$, then for the sequence of functions $(v_j)_{j\in\mathbb{Z}}$ satisfying (with some ball $\mathcal{B}$)

$$\mathrm{Supp}\,(\hat{v_j}) \subset 2^j\mathcal{B}, \quad \big(2^{js}\|v_j\|_{L^p}\big)_{j\in\mathbb{Z}} \in \ell^r(\mathbb{Z}), \quad \sum_{j\in\mathbb{Z}} v_j \to v \text{ in } \mathcal{S}' \text{ with } v \in \mathcal{S}'_h,$$

we have $v \in \dot B^s_{p,r}$ and

$$\|v\|_{\dot B^s_{p,r}} \leqslant C_s\Big\|\big(2^{js}\|v_j\|_{L^p}\big)_{j\in\mathbb{Z}}\Big\|_{\ell^r}.$$

Proof. By view of (2.5), the seminorm $\|\cdot\|_{\dot B^s_{p,r}}$ is indeed a norm. In order to show the density result, for any $u \in \dot B^s_{p,r}$, $p, r < \infty$, for any $\varepsilon > 0$, we take the approximated function

$$u^R_{M,N} = (\mathrm{Id} - \dot S_{-M})\big(\theta(\frac{\cdot}{R})u_N\big), \quad u_N := \sum_{|j|\leqslant N} \dot\Delta_j u,$$

where N is chosen such that

$$\|u - u_N\|_{\dot{B}^s_{p,r}} < \frac{\varepsilon}{2},$$

$M > N$, θ is a smooth cutoff function and R will be determined later. Then (**Exercise**) $u^R_{M,N} \in \mathcal{S}_0$ (while $u_N \in W^{\infty,p}(\mathbb{R}^d)$ is not necessarily in $\mathcal{S}$), and as $M > N$, we calculate

$$\begin{aligned}\|u^R_{M,N} - u_N\|_{\dot{B}^s_{p,r}} &= \Big\|(\mathrm{Id} - \dot{S}_{-M})\big(\theta(\frac{\cdot}{R}) - 1\big)u_N\Big\|_{\dot{B}^s_{p,r}} \\ &= \Big(\sum_{j\geqslant -M-1} 2^{jsr}\|\dot{\Delta}_j\big(\big(\theta(\frac{\cdot}{R}) - 1\big)u_N\big)\|^r_{L^p}\Big)^{\frac{1}{r}}.\end{aligned}$$

We decompose the summation $\sum_{j\geqslant -M-1}$ into high and low frequency parts separately again, such that (**Exercise: Check the following calculation**)

$$\begin{aligned}\sum_{j\geqslant 0} 2^{js}\|\dot{\Delta}_j\big(\big(\theta(\frac{\cdot}{R}) - 1\big)u_N\big)\|_{L^p} &\leqslant C\sup{}_{j\geqslant 0}2^{j([s]+2)}\|\dot{\Delta}_j\big(\big(\theta(\frac{\cdot}{R}) - 1\big)u_N\big)\|_{L^p} \\ &\leqslant C\|\nabla^{[s]+2}\big(\big(\theta(\frac{\cdot}{R}) - 1\big)u_N\big)\|_{L^p}\end{aligned}$$

$$\sum_{j=-M-1}^{-1} 2^{js}\|\dot{\Delta}_j\big(\big(\theta(\frac{\cdot}{R}) - 1\big)u_N\big)\|_{L^p} \leqslant C(M)\|\big(\theta(\frac{\cdot}{R}) - 1\big)u_N\|_{L^p},$$

and therefore for fixed M, N, as $p < \infty$, we can choose R large enough such that

$$\|u^R_{M,N} - u_N\|_{\dot{B}^s_{p,r}} < \frac{\varepsilon}{2},$$

and hence $\|u^R_{M,N} - u\|_{\dot{B}^s_{p,r}} < \varepsilon$.

We now show the independence of the choice of the function φ in the dyadic partition in the definition of the homogeneous Besov space $\dot{B}^s_{p,r}$. Indeed, if we take another function $\tilde{\varphi}$ in the dyadic partition, then there exists $J \in \mathbb{N}$ such that $\mathrm{Supp}\,\varphi(2^{-j}\cdot) \cap \mathrm{Supp}\,\tilde{\varphi}(2^{-j'}\cdot) = \varnothing$ if $|j - j'| \geqslant J$ and hence (**Exercise**) the norms $\|u\|_{\dot{B}^s_{p,r}}$ and $\|u\|_{\widetilde{\dot{B}^s_{p,r}}}$ are equivalent.

Similarly, if $\mathrm{Supp}\,(\hat{u}_j) \subset 2^j\mathcal{C}'$, then there exists $J \in \mathbb{N}$ such that

$$u_j = \sum_{|k-j|\leqslant J} \dot{\Delta}_k u_j \text{ and hence } \dot{\Delta}_k u = \sum_{|j-k|\leqslant J} \dot{\Delta}_k u_j.$$

Thus

$$\big\|(2^{ks}\|\dot{\Delta}_k u\|_{L^p})\big\|_{\ell^r} \leqslant \big\|\big(\sum_{|j-k|\leqslant J} 2^{(k-j)s}2^{js}\|u_j\|_{L^p}\big)_k\big\|_{\ell^r} \leqslant C(J,s)\big\|(2^{js}\|u_j\|_{L^p})_j\big\|_{\ell^r}.$$

If $\mathrm{Supp}\,(\hat{v}_j) \subset 2^j\mathcal{B}$, then there exists $J \in \mathbb{N}$ such that $\dot{\Delta}_k v = \sum_{j \geqslant k-J} \dot{\Delta}_k v_j$ and hence

$$\big\|(2^{ks}\|\dot{\Delta}_k v\|_{L^p})\big\|_{\ell^r} \leqslant \big\|(\sum_{j\geqslant k-J} 2^{(k-j)s}2^{js}\|v_j\|_{L^p})_k\big\|_{\ell^r} \leqslant C(J,s)\big\|(2^{js}\|v_j\|_{L^p})_j\big\|_{\ell^r},$$

where the last inequality is ensured by $s > 0$. □

Remark 2.2. *As $\mathcal{S}_0 \subset \mathcal{S} \subset \dot{B}^s_{p,r}$, $\mathcal{S}$ is also dense in $\dot{B}^s_{p,r}$ if $p < \infty, r < \infty$. If $s < \frac{d}{p}$ or if $(s,p,r) = (\frac{d}{p}, p, 1)$, then $\dot{B}^s_{p,r}(\mathbb{R}^d)$ can be a Banach space, while in other cases, due to the possible infrared divergence in the low frequency part (keeping in mind the polynomials), $\dot{B}^s_{p,r}(\mathbb{R}^d)$ is not a Banach space. Indeed, in general, if*

$$s < \frac{d}{p} \text{ or } (s,p,r) = (\frac{d}{p}, p, 1), \tag{2.7}$$

then by Lemma 1.2,

$$\lim_{j\to-\infty} \sum_{j'<j} \|\dot{\Delta}_{j'} u\|_{L^\infty} \leqslant \lim_{j\to-\infty} 2^{j(\frac{d}{p}-s)} \sum_{j'<j} \big(2^{js}\|\dot{\Delta}_{j'}u\|_{L^p}\big) = 0,$$

and hence $u \in \mathcal{S}'_h$ if $u \in \mathcal{S}'$ and $\|u\|_{\dot{B}^s_{p,r}} < \infty$.

Proposition 2.2. *We have the following basic properties for the homogeneous Besov spaces:*

- *Homogeneity: $\|u(\lambda\cdot)\|_{\dot{B}^s_{p,r}} = \lambda^{s-\frac{d}{p}}\|u\|_{\dot{B}^s_{p,r}}$, $\forall \lambda > 0$;*
- *Embedding: $\dot{B}^s_{p,r} \hookrightarrow \dot{B}^{s-d(\frac{1}{p}-\frac{1}{p_1})}_{p_1,r_1}$, if $p \leqslant p_1$, $r \leqslant r_1$ and in particular $\dot{B}^s_{p,1} \hookrightarrow \dot{B}^s_{p,r} \hookrightarrow \dot{B}^s_{p,\infty}$;*
- *Interpolation: $\dot{B}^{s_1}_{p,r} \cap \dot{B}^{s_2}_{p,r} \hookrightarrow \dot{B}^s_{p,r}$, $\forall s \in [s_1, s_2]$ and $\dot{B}^{s_1}_{p,\infty} \cap \dot{B}^{s_2}_{p,\infty} \hookrightarrow \dot{B}^s_{p,1}$, $\forall s \in (s_1, s_2)$. Furthermore, we have for any $u \in \mathcal{S}'_h$,*

$$\begin{aligned} &\|u\|_{\dot{B}^{\theta s_1+(1-\theta)s_2}_{p,r}} \leqslant \|u\|^\theta_{\dot{B}^{s_1}_{p,r}}\|u\|^{1-\theta}_{\dot{B}^{s_2}_{p,r}}, \quad \forall \theta \in [0,1]; \\ &\|u\|_{\dot{B}^{\theta s_1+(1-\theta)s_2}_{p,1}} \leqslant \frac{C}{s_2-s_1}\big(\frac{1}{\theta} + \frac{1}{1-\theta}\big)\|u\|^\theta_{\dot{B}^{s_1}_{p,\infty}}\|u\|^{1-\theta}_{\dot{B}^{s_2}_{p,\infty}}, \quad \forall \theta \in (0,1). \end{aligned} \tag{2.8}$$

Proof. The homogeneity property follows easily from the following calculation of change of variables (**Exercise**):

$$2^{js}\|\dot{\Delta}_j\big(u(\lambda x)\big)\|_{L^p} = \lambda^{(s-\frac{d}{p})}(2^j\lambda^{-1})^s\|(2^j\lambda^{-1})^d h(2^j\lambda^{-1}\cdot) * u\|_{L^p}.$$

where $h = (2\pi)^{-\frac{d}{2}}\check{\varphi}$.

We derive from Lemma 1.2 that if $p \leqslant p_1$, then

$$\|\dot\Delta_j u\|_{L^{p_1}} \leqslant C 2^{jd(\frac{1}{p}-\frac{1}{p_1})} \|\dot\Delta_j u\|_{L^p}.$$

This, together with $\|\cdot\|_{\ell^{r_1}} \leqslant \|\cdot\|_{\ell^r}$ if $r \leqslant r_1$, yields the embedding property $\dot B^s_{p,r} \hookrightarrow \dot B^{s-d(\frac{1}{p}-\frac{1}{p_1})}_{p_1,r_1}$.

The first inequality in (2.8) follows easily from

$$2^{j(\theta s_1+(1-\theta)s_2)}\|\dot\Delta_j u\|_{L^p} = \left(2^{js_1}\|\dot\Delta_j u\|_{L^p}\right)^\theta \left(2^{js_2}\|\dot\Delta_j u\|_{L^p}\right)^{1-\theta}$$

and Hölder's inequality. In order to show the second inequality in (2.8), we separate the high frequency and low frequency part in the definition of Besov norms:

$$\|u\|_{\dot B^{\theta s_1+(1-\theta)s_2}_{p,1}} = \sum_{j\leqslant N} 2^{j(\theta s_1+(1-\theta)s_2)}\|\dot\Delta_j u\|_{L^p} + \sum_{j>N} 2^{j(\theta s_1+(1-\theta)s_2)}\|\dot\Delta_j u\|_{L^p},$$

with the frequency threshold 2^N to be determined later. We use the Besov norm $\dot B^{s_1}_{p,\infty}$ to control the low frequency part:

$$\begin{aligned}\sum_{j\leqslant N} 2^{j(\theta s_1+(1-\theta)s_2)}\|\dot\Delta_j u\|_{L^p} &\leqslant \sum_{j\leqslant N} 2^{j(\theta s_1+(1-\theta)s_2)}\left(2^{-js_1}\|u\|_{\dot B^{s_1}_{p,\infty}}\right)\\ &= \sum_{j\leqslant N} 2^{j(1-\theta)(s_2-s_1)}\|u\|_{\dot B^{s_1}_{p,\infty}} = \frac{2^{N(1-\theta)(s_2-s_1)}}{2^{(1-\theta)(s_2-s_1)}-1}\|u\|_{\dot B^{s_1}_{p,\infty}},\end{aligned}$$

and similarly we use the Besov norm $\dot B^{s_2}_{p,\infty}$ to control the high frequency part:

$$\begin{aligned}\sum_{j>N} 2^{j(\theta s_1+(1-\theta)s_2)}\|\dot\Delta_j u\|_{L^p} &\leqslant \sum_{j>N} 2^{j(\theta s_1+(1-\theta)s_2)}\left(2^{-js_2}\|u\|_{\dot B^{s_2}_{p,\infty}}\right)\\ &= \sum_{j>N} 2^{-j\theta(s_2-s_1)}\|u\|_{\dot B^{s_2}_{p,\infty}} \leqslant \frac{2^{-N\theta(s_2-s_1)}}{1-2^{-\theta(s_2-s_1)}}\|u\|_{\dot B^{s_2}_{p,\infty}}.\end{aligned}$$

We choose N such that

$$2^{N(s_2-s_1)} = \frac{\|u\|_{\dot B^{s_2}_{p,\infty}}}{\|u\|_{\dot B^{s_1}_{p,\infty}}}$$

such that the second inequality in (2.8) holds. □

[13.05.2019]

[20.05.2019]

We define the homogeneous Sobolev spaces $\dot H^s$ as follows:

Definition 2.3. *Let $s \in \mathbb{R}$. We denote by $\dot{H}^s(\mathbb{R}^d)$ the set of tempered distributions $u \in \mathcal{S}'$ such that $\hat{u} \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and*

$$\|u\|_{\dot{H}^s} = \Big(\int_{\mathbb{R}^d} |\xi|^{2s} |\hat{u}(\xi)|^2 \,\mathrm{d}\xi\Big)^{\frac{1}{2}} < \infty.$$

Then $\dot{H}^s \subset \mathcal{S}'_h$. Since $\|\|_{\dot{H}^s} \sim \|\|_{\dot{B}^s_{2,2}}$, $\dot{H}^s \subset \dot{B}^s_{2,2}$ for any $s \in \mathbb{R}$. If $s < \frac{d}{2}$ (i.e. (2.7) holds), then $\dot{H}^s = \dot{B}^s_{2,2}$.

We also have the following relations between the Besov spaces and the Lebesgue spaces:

Proposition 2.3. *Let $(p,q) \in [1,\infty]^2$. Then*

- $\dot{B}^0_{p,1} \hookrightarrow L^p \hookrightarrow \dot{B}^0_{p,\infty}$ *and more generally* $\dot{B}^{\frac{d}{p}-\frac{d}{q}}_{p,1} \hookrightarrow L^q$ *whenever* $p \leqslant q$*;*
- $\dot{B}^0_{p,2} \hookrightarrow L^p \hookrightarrow \dot{B}^0_{p,p}$ *whenever* $p \in [2,\infty)$ *and* $\dot{B}^0_{p,p} \hookrightarrow L^p \hookrightarrow \dot{B}^0_{p,2}$ *whenever* $p \in (1,2]$*, and in particular* $L^2 = \dot{B}^0_{2,2}$*;*
- $\dot{B}^{\frac{d}{p}}_{p,1} \hookrightarrow C_0$ *whenever* $p \in [1,\infty)$*;*
- *The space of bounded measures on $\mathbb{R}^d$ is continuously embedded in $\dot{B}^0_{1,\infty}$;*
- $\|u\|_{L^q} \leqslant C\|u\|^{1-\theta}_{\dot{B}^{-\alpha}_{\infty,\infty}} \|u\|^{\theta}_{\dot{B}^{\beta}_{p,p}}$ *whenever* $1 \leqslant p < q < \infty$*,* $\alpha \in \mathbb{R}^+$ *with* $\theta = \frac{p}{q}$ *and* $\beta = \alpha(\frac{q}{p} - 1)$*.*

We may also define the homogeneous Triebel-Lizorkin space $\dot{F}^s_{p,r}(\mathbb{R}^d)$, $1 < p < \infty$ as

$$\dot{F}^s_{p,r}(\mathbb{R}^d) = \{u \in \mathcal{S}'_h \mid \|u\|_{\dot{F}^s_{p,r}} = \Big\| \big\| \big(2^{js}|\dot{\Delta}_j u(x)|\big)_{j\in\mathbb{Z}} \big\|_{\ell^r} \Big\|_{L^p} < \infty\},$$

and the Lorentz space $L^{p,r}(\mathbb{R}^d)$, $1 \leqslant p < \infty$ as

$$L^{p,r}(\mathbb{R}^d) = \{u : \mathbb{R}^d \mapsto \mathbb{C} \mid \|u\|_{L^{p,r}} = \big\|s^{\frac{1}{r}} u^*(s)\big\|_{L^r(\mathbb{R}^+;\frac{ds}{s})} < \infty\},$$

where u^* is the rearrangement function of u:

$$u^*(s) = \inf\{\lambda \mid m(\{x \in \mathbb{R}^d \mid |u(x)| \geqslant \lambda\}) \leqslant s\}.$$

There are relations among the above mentioned functional spaces which we do not go to details here.

2.2 Homogeneous paradifferential calculus

Let $u, v \in \mathcal{S}'_h$ and we would like to consider their product uv (which is in general not well defined). Recall the Littlewood-Paley decomposition (2.4)-(2.5):

$$u = \sum_{j\in\mathbb{Z}} \dot{\Delta}_j u \text{ and } \lim_{j\to-\infty} \dot{S}_j u = 0 \text{ in } \mathcal{S}',$$
$$v = \sum_{j\in\mathbb{Z}} \dot{\Delta}_j v \text{ and } \lim_{j\to-\infty} \dot{S}_j v = 0 \text{ in } \mathcal{S}',$$

where

$$\dot{\Delta}_j u = \varphi(2^{-j}D)u \text{ with } \widehat{\varphi(2^{-j}D)u} = \varphi(2^{-j}\xi)\hat{u}(\xi), \quad j \in \mathbb{Z},$$
$$\dot{S}_j u = \chi(2^{-j}D)u \text{ with } \widehat{\chi(2^{-j}D)u} = \chi(2^{-j}\xi)\hat{u}(\xi), \quad j \in \mathbb{Z}\,.$$

Then formally we can write the product uv as

$$uv = \sum_{j,k\in\mathbb{Z}} \dot{\Delta}_j u \dot{\Delta}_k v,$$

and we can split the above sum into three parts:

$$uv = \sum_{j\in\mathbb{Z}} \dot{S}_{j-1}u\dot{\Delta}_j v + \sum_{j\in\mathbb{Z}} \dot{\Delta}_j u \dot{S}_{j-1} v + \sum_{|j-k|\leqslant 1} \dot{\Delta}_j u \dot{\Delta}_k v.$$

Notice that the Fourier transform of $\dot{S}_{j-1}u = \chi(2^{-(j-1)}D)u$ is compactly supported on the ball of size $\frac{4}{3}2^{(j-1)}$ and the Fourier transform of $\dot{\Delta}_j u = \varphi(2^{-j}D)u$ is compactly supported on the annulus $\{\xi \in \mathbb{R}^d \mid 2^j \leqslant |\xi| \leqslant \frac{8}{3}2^j\}$, such that

$$\widehat{\dot{S}_{j-1}u\dot{\Delta}_j v} = (2\pi)^{-\frac{d}{2}} \widehat{\dot{S}_{j-1}u} * \widehat{\dot{\Delta}_j v}, \quad j \in \mathbb{Z}$$

is compactly supported on the annulus $\{\xi \in \mathbb{R}^d \mid \frac{1}{3}2^j \leqslant |\xi| \leqslant \frac{10}{3}2^j\}$, while

$$\widehat{\dot{\Delta}_j u \dot{\Delta}_k v} = (2\pi)^{-\frac{d}{2}} \widehat{\dot{\Delta}_j u} * \widehat{\dot{\Delta}_k v}, \quad |j-k| \leqslant 1$$

is compactly supported on the ball of size $8 \cdot 2^j$.

Definition 2.4. *Let $u, v \in \mathcal{S}'_h$. We denote by $\dot{T}_u v$ the homogeneous paraproduct of v by u as follows:*

$$\dot{T}_u v = \sum_{j\in\mathbb{Z}} \dot{S}_{j-1}u\dot{\Delta}_j v.$$

We denote by $\dot{R}(u,v)$ the homogeneous remainder of u and v as follows:

$$\dot{R}(u,v) = \sum_{|j-k|\leqslant 1} \dot{\Delta}_j u \dot{\Delta}_k v.$$

We call the above product decomposition of uv the Bony decomposition:

$$uv = \dot{T}_u v + \dot{T}_v u + \dot{R}(u,v).$$

The summations in the above definitions of the bilinear operators $\dot{T}, \dot{R}$ are formal and we are going to make it rigorous in the setting of homogeneous Besov spaces, by use of the above consideration of the supports of the Fourier transforms and Theorem 2.1.

Theorem 2.2. *Let $(s,p,r) \in \mathbb{R} \times [1,\infty]^2$ satisfy (2.7): $s < \frac{d}{p}$ or $(s,p,r) = (\frac{d}{p}, p, 1)$. Then there exists C (depending only on s) such that $\dot{T}_u v \in \mathcal{S}'_h$ and*

$$\|\dot{T}_u v\|_{\dot{B}^s_{p,r}} \leqslant C\|u\|_{L^\infty}\|v\|_{\dot{B}^s_{p,r}}.$$

If furthermore $\sigma < 0$, then there exists C (depending only on $|s+\sigma|$) such that

$$\|\dot{T}_u v\|_{\dot{B}^{s+\sigma}_{p,r}} \leqslant C\|u\|_{\dot{B}^\sigma_{\infty,r_1}}\|v\|_{\dot{B}^s_{p,r_2}}, \text{ with } \frac{1}{r} = \min\{1, \frac{1}{r_1} + \frac{1}{r_2}\}.$$

Proof. Since $\dot{T}_u v = \sum_{j\in\mathbb{Z}} \dot{S}_{j-1} u \dot{\Delta}_j v$ with $\widehat{\dot{S}_{j-1} u \dot{\Delta}_j v}$ compactly supported on the annulus $\{\xi \in \mathbb{R}^d \mid \frac{1}{3} 2^j \leqslant |\xi| \leqslant \frac{10}{3} 2^j\}$, we apply Theorem 2.1 and Bernstein's inequality to derive

$$\begin{aligned}\|\dot{T}_u v\|_{\dot{B}^s_{p,r}} &\leqslant C\big\|\big(2^{js}\|\dot{S}_{j-1} u \dot{\Delta}_j v\|_{L^p}\big)\big\|_{\ell^r} \leqslant C\big\|\big(2^{js}\|\dot{S}_{j-1} u\|_{L^\infty}\|\dot{\Delta}_j v\|_{L^p}\big)\big\|_{\ell^r} \\ &\leqslant C\sup{}_{j\in\mathbb{Z}}\|\dot{S}_{j-1} u\|_{L^\infty}\big\|\big(2^{js}\|\dot{\Delta}_j v\|_{L^p}\big)\big\|_{\ell^r} \leqslant C\|u\|_{L^\infty}\|v\|_{\dot{B}^s_{p,r}}.\end{aligned}$$

We claim (**Exercise**) that if $\sigma < 0$, then

$$\Big\|\big(2^{j\sigma}\|\dot{S}_j u\|_{L^p}\big)\Big\|_{\ell^r} \leqslant C\|u\|_{\dot{B}^\sigma_{p,r}}, \tag{2.9}$$

and hence the same argument as above yields

$$\|\dot{T}_u v\|_{\dot{B}^{s+\sigma}_{p,r}} \leqslant C\big\|\big(2^{j\sigma}\|\dot{S}_{j-1} u\|_{L^\infty}\big)\big\|_{\ell^{r_1}}\big\|\big(2^{js}\|\dot{\Delta}_j v\|_{L^p}\big)\big\|_{\ell^{r_2}} = C\|u\|_{\dot{B}^\sigma_{\infty,r_1}}\|v\|_{\dot{B}^s_{p,r_2}}.$$

□

Theorem 2.3. *Let $(s,\sigma)\in\mathbb{R}^2$ with $s+\sigma>0$. Let $(s+\sigma,p,r)\in\mathbb{R}^+\times[1,\infty]^2$ satisfy (2.7), $(p_1,p_2)\in[1,\infty]^2$ satisfy $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$ and (r_1,r_2) satisfy $\frac{1}{r}=\min(1,\frac{1}{r_1}+\frac{1}{r_2})$. Then there exists C (depending only on $s+\sigma$) such that*

$$\|\dot{R}(u,v)\|_{\dot{B}^{s+\sigma}_{p,r}}\leqslant C\|u\|_{\dot{B}^{s}_{p_1,r_1}}\|v\|_{\dot{B}^{\sigma}_{p_2,r_2}}.$$

Proof. Since $s+\sigma>0$ and $\dot{R}(u,v)=\sum_{j\in\mathbb{Z}}(\sum_{|j-k|\leqslant1}\dot{\Delta}_ju\dot{\Delta}_kv)$ with $\widehat{\sum_{|j-k|\leqslant1}\dot{\Delta}_ju\dot{\Delta}_kv}$ compactly supported on the ball of size $8\cdot2^j$, we derive from Theorem 2.1 that

$$\begin{aligned}\|\dot{R}(u,v)\|_{\dot{B}^{s+\sigma}_{p,r}}&\leqslant C\Big\|\big(2^{j(s+\sigma)}\|\sum_{|j-k|\leqslant1}\dot{\Delta}_ju\dot{\Delta}_kv\|_{L^p}\big)\Big\|_{\ell^r}\\&\leqslant C\Big\|\big(2^{js}\|\dot{\Delta}_ju\|_{L^{p_1}}\big)\Big\|_{\ell^{r_1}}\Big\|\big(2^{j\sigma}\sum_{|j-k|\leqslant1}\|\dot{\Delta}_kv\|_{L^{p_2}}\big)\Big\|_{\ell^{r_2}}\leqslant C\|u\|_{\dot{B}^{s}_{p_1,r_1}}\|v\|_{\dot{B}^{\sigma}_{p_2,r_2}}.\end{aligned}$$

□

Remark 2.3. *If $s+\sigma=0$ and $r=1$, then from the above proof we have*

$$\|\dot{R}(u,v)\|_{\dot{B}^{0}_{p,\infty}}\leqslant C\|u\|_{\dot{B}^{s}_{p_1,r_1}}\|v\|_{\dot{B}^{-s}_{p_2,r_2}}.$$

Corollary 2.1. *Let $(s,p,r)\in\mathbb{R}^+\times[1,\infty]^2$ satisfy (2.7). Then $L^\infty\cap\dot{B}^s_{p,r}$ is an algebra and*

$$\|uv\|_{\dot{B}^{s}_{p,r}}\leqslant C(\|u\|_{L^\infty}\|v\|_{\dot{B}^{s}_{p,r}}+\|u\|_{\dot{B}^{s}_{p,r}}\|v\|_{L^\infty}).$$

Proof. We simply derive from Theorem 2.2 and Theorem 2.3 that

$$\begin{aligned}\|uv\|_{\dot{B}^{s}_{p,r}}&\leqslant\|\dot{T}_uv\|_{\dot{B}^{s}_{p,r}}+\|\dot{T}_vu\|_{\dot{B}^{s}_{p,r}}+\|\dot{R}(u,v)\|_{\dot{B}^{s}_{p,r}}\\&\leqslant C\|u\|_{L^\infty}\|v\|_{\dot{B}^{s}_{p,r}}+C\|v\|_{L^\infty}\|u\|_{\dot{B}^{s}_{p,r}}+C\|u\|_{\dot{B}^{0}_{\infty,\infty}}\|v\|_{\dot{B}^{s}_{p,r}},\end{aligned}$$

which, together with the fact $\|u\|_{\dot{B}^{0}_{\infty,\infty}}=\sup_{j\in\mathbb{Z}}\|\dot{\Delta}_ju\|_{L^\infty}\leqslant C\|u\|_{L^\infty}$, implies the result. □

[20.05.2019]

[21.05.2019]

Corollary 2.2. *Let $(s,\sigma)\in(-\frac{d}{2},\frac{d}{2})^2$ such that $s+\sigma>0$, then*

$$\|uv\|_{\dot{B}^{s+\sigma-\frac{d}{2}}_{2,1}}\leqslant C\|u\|_{\dot{H}^s}\|v\|_{\dot{H}^\sigma}.$$

Proof. Since $\dot{H}^s = \dot{B}^s_{2,2} \hookrightarrow \dot{B}^{s-\frac{d}{2}}_{\infty,2}$, we derive from Theorem 2.2 that

$$\|\dot{T}_u v\|_{\dot{B}^{s+\sigma-\frac{d}{2}}_{2,1}} \leqslant C\|u\|_{\dot{B}^{s-\frac{d}{2}}_{\infty,2}}\|v\|_{\dot{B}^{\sigma}_{2,2}} \leqslant C\|u\|_{\dot{H}^s}\|v\|_{\dot{H}^\sigma},$$

and similarly we have the above inequality for $\dot{T}_v u$. We now apply Theorem 2.3 and the embedding $\dot{B}^{s+\sigma}_{1,1} \hookrightarrow \dot{B}^{s+\sigma-\frac{d}{2}}_{2,1}$ to $\dot{R}(u,v)$ to arrive at

$$\|\dot{R}(u,v)\|_{\dot{B}^{s+\sigma-\frac{d}{2}}_{2,1}} \leqslant \|\dot{R}(u,v)\|_{\dot{B}^{s+\sigma}_{1,1}} \leqslant C\|u\|_{\dot{B}^s_{2,2}}\|v\|_{\dot{B}^\sigma_{2,2}}.$$

□

Corollary 2.3 (Hardy's inequality)**.** *Let $s \in [0, \frac{d}{2})$, then there exists C such that*

$$\int_{\mathbb{R}^d} \frac{|f(x)|^2}{|x|^{2s}}\,\mathrm{d}x \leqslant C\|f\|^2_{\dot{H}^s}.$$

Proof. **Exercise.** Notice that

$$\int_{\mathbb{R}^d} \frac{|f(x)|^2}{|x|^{2s}}\,\mathrm{d}x \leqslant \sum_{|j-k|\leqslant 1} \|\dot{\Delta}_j(|f|^2)\|_{L^2}\|\dot{\Delta}_k(|x|^{-2s})\|_{L^2} \leqslant C\||f|^2\|_{\dot{B}^{2s-\frac{d}{2}}_{2,1}}\||x|^{-2s}\|_{\dot{B}^{\frac{d}{2}-2s}_{2,\infty}}.$$

□

Remark 2.4. *If $d \geqslant 3$ and $s = 1$, then we simply have*

$$\int_{\mathbb{R}^d} \frac{|f(x)|^2}{|x|^2}\,\mathrm{d}x \leqslant C\|\nabla f\|^2_{L^2}.$$

2.3 Nonhomogeneous Besov spaces

In this subsection we give the definition of the nonhomogeneous Besov space. Most of the results in Subsection 2.1 still hold true in the nonhomogeneous framework, and the proofs are even simpler as we take the Littlewood-Paley decomposition (2.2): $u = \Delta_{-1}u + \sum_{j\geqslant 0}\Delta_j u$ and we do not have to worry about the low frequencies $\dot{\Delta}_j u$, $j \leqslant -1$.

Definition 2.5. *Let $s \in \mathbb{R}$, $(p,r) \in [1,\infty]^2$. The nonhomogeneous Besov space $B^s_{p,r}$ consists of all tempered distributions $u \in \mathcal{S}'(\mathbb{R}^d)$ such that*

$$\|u\|_{B^s_{p,r}} := \left\|\left(2^{js}\|\Delta_j u\|_{L^p}\right)_{j\geqslant -1}\right\|_{\ell^r} < \infty.$$

Remark 2.5. *When $p = r = 2$, then*

$$B^s_{2,2} = H^s := \{u \in \mathcal{S}' \mid \|u\|^2_{H^s} := \int_{\mathbb{R}^d} (1 + |\xi|^2)^s |\hat{u}|^2 \,\mathrm{d}\xi \ < \infty\}.$$

When $s \in (0,1)$, $p = r = \infty$, then

$$B^s_{\infty,\infty} := C^s = \{u \in C_b(\mathbb{R}^d) \mid [u]_s := \sup{}_{x \neq y} \frac{|u(x) - u(y)|}{|x-y|^s} < \infty\}.$$

We just list the properties that still hold true in the nonhomogeneous setting and the proofs are left to the interested readers:

- Theorem 2.1 holds in the nonhomogeneous setting except that we now have the fact that $\mathcal{D}(\mathbb{R}^d)$ is dense in $B^s_{p,r}(\mathbb{R}^d)$ when $p < \infty$ and $r < \infty$. Furthermore, $B^s_{p,r}(\mathbb{R}^d)$ is a Banach space.
- The embedding property and interpolation property in Proposition 2.2 hold true in the nonhomogeneous setting.
- Theorem 2.2 and Theorem 2.3 hold in the nonhomogeneous setting, where the nonhomogeneous paraproduct is defined by

  $$T_u v = \sum_j S_{j-1} u \Delta_j v, \quad S_{j-1} u = \sum_{-1 \leqslant k \leqslant j-2} \Delta_k u,$$

  and the nonhomgeneous remainder is defined by

  $$R(u,v) = \sum_{|k-j| \leqslant 1} \Delta_j u \Delta_k v.$$

  Correspondingly Corollary 2.1 and Corollary 2.2 hold true in the nonhomogeneous setting.

2.4 Commutator estimates

Let us consider the transport equation

$$\begin{cases} \partial_t \rho + v \cdot \nabla \rho = f, \quad t \in \mathbb{R}, \ x \in \mathbb{R}^d, \\ \rho|_{t=0} = \rho_0, \end{cases} \tag{2.10}$$

where the unknown function $\rho = \rho(t,x) \in \mathbb{R}$ is transported by the velocity vector field $v = v(t,x) \in \mathbb{R}^d$ and $f = f(t,x) \in \mathbb{R}$ is the source term. We apply the operator Δ_j to it to arrive at the transport equation for $\Delta_j \rho$:

$$\begin{cases} \partial_t (\Delta_j \rho) + v \cdot \nabla (\Delta_j \rho) = \Delta_j f + [v \cdot \nabla, \Delta_j] \rho, \\ (\Delta_j \rho)|_{t=0} = \Delta_j \rho_0, \end{cases} \tag{2.11}$$

where $[v\cdot\nabla,\Delta_j]\rho$ denotes the commutator $v\cdot\nabla(\Delta_j\rho)-\Delta_j(v\cdot\nabla\rho)$. In order to transport the $B^s_{p,r}$-regularity of the unknown function ρ, we have to estimate the commutator as follows:

$$\Big\|\big(2^{js}\|[v\cdot\nabla,\Delta_j]\rho\|_{L^p}\big)_{j\geqslant -1}\Big\|_{\ell^r}.$$

Theorem 2.4. *Let $s\in\mathbb{R}$, $(p,r)\in[1,\infty]^2$ and $v:\mathbb{R}^d\mapsto\mathbb{R}^d$ be a vector field. Assume further that*

$$\begin{aligned} &s>\max\{-\frac{d}{p},-\frac{d}{p'}\}\ \textit{with}\ \frac{1}{p'}=1-\frac{1}{p},\\ &\textit{or}\ s>-1+\max\{-\frac{d}{p},-\frac{d}{p'}\}\ \textit{if}\ \operatorname{div}v:=\sum_{j=1}^d\partial_{x_j}v_j=0. \end{aligned}\tag{2.12}$$

Then there exists a constant C such that

$$\Big\|\big(2^{js}\|[v\cdot\nabla,\Delta_j]\rho\|_{L^p}\big)_{j\geqslant -1}\Big\|_{\ell^r}\leqslant C\|\nabla v\|_{B^{\frac{d}{p}}_{p,\infty}\cap L^\infty}\|\rho\|_{B^s_{p,r}}\ \textit{if}\ s<1+\frac{d}{p}.\tag{2.13}$$

For general $s>0$ or $s>-1$ when $\operatorname{div}v=0$, then

$$\Big\|\big(2^{js}\|[v\cdot\nabla,\Delta_j]\rho\|_{L^p}\big)_{j\geqslant -1}\Big\|_{\ell^r}\leqslant C\big(\|\nabla v\|_{L^\infty}\|\rho\|_{B^s_{p,r}}+\|\nabla v\|_{B^{s-1}_{p,r}}\|\nabla\rho\|_{L^\infty}\big).\tag{2.14}$$

Proof. We decompose v into the low frequency and high frequency parts:

$$v=\Delta_{-1}v+\tilde v,\quad \tilde v=\sum_{j\geqslant 0}\Delta_j v.$$

To warm up, we have (**Exercise**) for the low frequency part that

$$\Big\|\big(2^{js}\|[\Delta_{-1}v\cdot\nabla,\Delta_j]\rho\|_{L^p}\big)_{j\geqslant -1}\Big\|_{\ell^r}\leqslant C\|\nabla v\|_{L^\infty}\|\nabla\rho\|_{B^{s-1}_{p,r}},\tag{2.15}$$

by calculating straightforward

$$\begin{aligned}[]
[\Delta_{-1}v\cdot\nabla,\Delta_j]\rho &= (2\pi)^{-\frac{d}{2}}\int_{\mathbb{R}^d}\check\varphi_j(x-y)\big((\Delta_{-1}v)(x)-(\Delta_{-1}v)(y)\big)\cdot\nabla\rho(y)\,\mathrm{d}y\\
&=(2\pi)^{-\frac{d}{2}}\int_{\mathbb{R}^d}\check\varphi_j(x-y)\Big((x-y)\cdot\int_0^1(\nabla\Delta_{-1}v)(x-\tau(x-y))d\tau\Big)\cdot\nabla\rho(y)\,\mathrm{d}y
\end{aligned}$$

and noticing

$$[\Delta_{-1}v\cdot\nabla,\Delta_j]\rho=\sum_{|j'-j|\leqslant 1}[\Delta_{-1}v\cdot\nabla,\Delta_j](\Delta_{j'}\rho)$$

and $\check\varphi_j(x-y)\,(x-y)=2^{jd}\check\varphi(2^j(x-y))\,(x-y)=2^{-j}\Big(2^{jd}\big(z\check\varphi(z)\big)|_{z=2^j(x-y)}\Big)$.

Now we consider the commutator

$$[\tilde v\cdot\nabla,\Delta_j]\rho=\sum_{k=1}^d\big(\tilde v^k\partial_k\Delta_j\rho-\Delta_j(\tilde v^k\partial_k\rho)\big)$$

$$=\sum_{k=1}^d T_{\tilde v^k}\partial_k(\Delta_j\rho)+T_{\partial_k\Delta_j\rho}\tilde v^k+R(\tilde v^k,\partial_k\Delta_j\rho)-\Delta_j(T_{\tilde v^k}\partial_k\rho+T_{\partial_k\rho}\tilde v^k+R(\tilde v^k,\partial_k\rho))$$

$$=\sum_{k=1}^d[T_{\tilde v^k},\Delta_j]\partial_k\rho+T_{\partial_k\Delta_j\rho}\tilde v^k-\Delta_j(T_{\partial_k\rho}\tilde v^k)+R(\tilde v^k,\partial_k\Delta_j\rho)-\Delta_jR(\tilde v^k,\partial_k\rho),$$

and we are going to estimate the terms one by one.

Similarly as above (**Exercise**), $[T_{\tilde v^k},\Delta_j]\partial_k\rho=\sum_{|j-j'|\leqslant 4}[S_{j'-1}\tilde v^k,\Delta_j]\partial_k\Delta_{j'}\rho$ satisfies (2.15).

[21.05.2019]

[27.05.2019]

As

$$\begin{aligned}\|T_{\partial_k\Delta_j\rho}\tilde v^k\|_{L^p}&=\Big\|\sum_{j'\geqslant j-3}S_{j'-1}\partial_k\Delta_j\rho\,\Delta_{j'}\tilde v^k\Big\|_{L^p}\\&\leqslant C\|\partial_k\Delta_j\rho\|_{L^p}\sum_{j'\geqslant j-3}\|\Delta_{j'}\tilde v^k\|_{L^\infty}\\&\leqslant C\|\Delta_j(\partial_k\rho)\|_{L^p}\sum_{j'\geqslant j-3}2^{-j'}\|\Delta_{j'}(\nabla\tilde v^k)\|_{L^\infty}\\&\leqslant C\|\Delta_j(\partial_k\rho)\|_{L^p}2^{-j}\|\nabla v\|_{L^\infty},\end{aligned}$$

(2.15) follows for the term $T_{\partial_k\Delta_j\rho}\tilde v^k$.

Now we come to

$$\big\|\Delta_j(T_{\partial_k\rho}\tilde v^k)\big\|_{L^p}=\Big\|\sum_{|j-j'|\leqslant 4}\Delta_j(S_{j'-1}\partial_k\rho\,\Delta_{j'}\tilde v^k)\Big\|_{L^p}\leqslant\sum_{|j-j'|\leqslant 4}\|S_{j'-1}\partial_k\rho\|_{L^\infty}\|\Delta_{j'}\tilde v^k\|_{L^p}.$$

Recall the characterisation (2.9) of $B^{s-1-\frac dp}_{\infty,r}$ when $s<1+\frac dp$, such that

$$\begin{aligned}&\Big\|\big(2^{js}\|\Delta_j(T_{\partial_k\rho}\tilde v^k)\|_{L^p}\big)_{j\geqslant-1}\Big\|_{\ell^r}\\&\leqslant C\Big\|\big(2^{j(s-1-\frac dp)}\|S_{j-1}\nabla\rho\|_{L^\infty}\big)_{j\geqslant-1}\Big\|_{\ell^r}\Big\|\big(2^{j(1+\frac dp)}\|\Delta_j\tilde v^k\|_{L^p}\big)_{j\geqslant-1}\Big\|_{\ell^\infty}\\&\leqslant C\|\nabla\rho\|_{B^{s-1-\frac dp}_{\infty,r}}\|\tilde v\|_{B^{1+\frac dp}_{p,\infty}}\leqslant C\|\nabla\rho\|_{B^{s-1}_{p,r}}\|\nabla v\|_{B^{\frac dp}_{p,\infty}}.\end{aligned}$$

We can also simply estimate the above as (without any restriction on s):

$$\begin{aligned}\Big\|\big(2^{js}\|\Delta_j(T_{\partial_k\rho}\tilde v^k)\|_{L^p}\big)_{j\geqslant-1}\Big\|_{\ell^r}&\leqslant C\|\nabla\rho\|_{L^\infty}\Big\|\big(2^{js}\|\Delta_j\tilde v^k\|_{L^p}\big)_{j\geqslant-1}\Big\|_{\ell^r}\\&\leqslant C\|\nabla\rho\|_{L^\infty}\|\nabla v\|_{B^{s-1}_{p,r}}.\end{aligned}$$

Next it is straightforward to estimate

$$\Big\|\big(2^{js}\|R(\tilde v^k,\partial_k\Delta_j\rho)\|_{L^p}\big)_{j\geqslant -1}\Big\|_{\ell^r} = \Big\|\big(2^{js}\|\sum_{|j-j'|\leqslant 2}\Delta_{j'}\tilde v^k\partial_k\Delta_j(\sum_{|j''-j'|\leqslant 1}\Delta_{j''}\rho)\|_{L^p}\big)_{j\geqslant -1}\Big\|_{\ell^r}$$
$$\leqslant C\|\nabla v\|_{L^\infty}\|\nabla\rho\|_{B^{s-1}_{p,r}}.$$

Finally we can follow the idea in the proof of Theorem 2.3 to control (**Exercise**)

$$\Big\|\big(2^{js}\|\Delta_j\sum_{k=1}^d R(\tilde v^k,\partial_k\rho)\|_{L^p}\big)_{j\geqslant -1}\Big\|_{\ell^r}$$
$$=\Big\|\big(2^{js}\|\sum_{k=1}^d \Delta_j\partial_k R(\tilde v^k,\rho)-\Delta_j R(\operatorname{div}\tilde v,\rho)\|_{L^p}\big)_{j\geqslant -1}\Big\|_{\ell^r}.$$

Indeed, in the general case where we do not know $\operatorname{div} v = 0$, we simply rewrite

$$2^{js}\|\Delta_j R(\tilde v^k,\partial_k\rho)\|_{L^p}\leqslant C\begin{cases} 2^{j(s+\frac{d}{p})}\|\Delta_j R(\tilde v^k,\partial_k\rho)\|_{L^{\frac{p}{2}}} & \text{if } s>-\frac{d}{p}\geqslant-\frac{d}{p'},\ \text{i.e. } p\geqslant 2,\\ 2^{j(s+\frac{d}{p'})}\|\Delta_j R(\tilde v^k,\partial_k\rho)\|_{L^1} & \text{if } s>-\frac{d}{p'}\geqslant-\frac{d}{p},\ \text{i.e. } p\leqslant 2,\end{cases}$$

then we make use of $\Delta_j R(\tilde v^k,\partial_k\rho)=\sum_{j'\geqslant j-4}\sum_{|j'-j''|\leqslant 1}\Delta_j\big((\Delta_{j'}\tilde v^k)(\Delta_{j''}\partial_k\rho)\big)$ to derive (2.13). If $\operatorname{div} v = 0$ such that $\operatorname{div}\tilde v = -\operatorname{div}\Delta_{-1}v$, then under the assumption $s>-1+\max\{-\frac{d}{p},-\frac{d}{p'}\}$ we still have (2.13). Under the assumption $s>0$ or $s>-1$ if $\operatorname{div} v=0$, the inequality (2.14) follows similarly. □

3 Applications to PDEs

3.1 Transport equation

In this section we will consider the transport equation (2.10):

$$\begin{cases}\partial_t\rho+v\cdot\nabla\rho=f, & t\in\mathbb{R},\ x\in\mathbb{R}^d,\\ \rho|_{t=0}=\rho_0,\end{cases}\tag{T}$$

where the function $\rho=\rho(t,x)\in\mathbb{R}$ is transported by the velocity vector field $v=v(t,x)\in\mathbb{R}^d$ and $f=f(t,x)\in\mathbb{R}$ is the source term.

We will first briefly revisit the classical Cauchy-Lipschitz theorem in the ODE theory, and then we will solve (T) by use of the commutator estimate established in Subsection 2.4.

3.1.1 The Cauchy-Lipschitz theorem

Theorem 3.1. *Let E be a Banach space, $\Omega \subset E$ an open set, $I \ni 0$ an open time interval and $X_0 \in \Omega$.*

Let $v : I \times \Omega \mapsto E$ be $L^1_{\mathrm{loc}}(I; \mathrm{Lip}(\Omega; E))$ in the following sense:

$$\int_K \sup_{\{(X_1(t),X_2(t))\in\Omega^2 \,|\, X_1(t)\neq X_2(t)\}} \frac{\|v(t,X_1(t)) - v(t,X_2(t))\|_E}{\|X_1(t) - X_2(t)\|_E} dt < \infty, \quad \forall K \text{ compact set in } I.$$

Then there exists an open time interval $J \ni 0$ (with $J \subset I$) such that the equation

$$X(t) = X_0 + \int_0^t v(t', X(t'))\,\mathrm{d}t'\,, \tag{ODE}$$

i.e. $\frac{d}{dt}X(t) = v(t, X(t))$ with $X(0) = X_0$ has a unique continuous solution $X = X(t) : J \mapsto \Omega$.

Proof. (**Exercise**) The theorem follows from the Picard iteration scheme:

$$X_{k+1}(t) = X_0 + \int_0^t v(t', X_k(t'))\,\mathrm{d}t'\,, \quad \forall k \geqslant 0.$$

Indeed, since

$$\begin{aligned}
\|X_{k+1} - X_k\|_E &\leqslant \int_0^t \|v(t', X_k(t')) - v(t', X_{k-1}(t'))\|_E\,\mathrm{d}t' \\
&\leqslant \int_0^t \|X_k(t') - X_{k-1}(t')\|_E \gamma(t')\,\mathrm{d}t'\,,
\end{aligned}$$

where the function $\gamma = \gamma(t)$ characterizes the Lipschitz dependence of the function $v = v(t, \cdot)$:

$$\gamma(t) := \sup_{\{(X_1(t),X_2(t))\in\Omega^2 \,|\, X_1(t)\neq X_2(t)\}} \frac{\|v(t,X_1(t)) - v(t,X_2(t))\|_E}{\|X_1(t) - X_2(t)\|_E},$$

then there exists some time interval $J \ni 0$ such that the sequence $X_0 + \sum_{k\geqslant 0}(X_{k+1}(t) - X_k(t))$ converges to $X(t)$ in $\Omega \subset E$ uniformly on J by virtue of

$$\sup_{t\in J} \|X_{k+1}(t) - X_k(t)\|_E \leqslant \sup_{t\in J} \|X_k(t) - X_{k-1}(t)\|_E \int_J \gamma(t')\,\mathrm{d}t'$$

and $\gamma \in L^1_{\mathrm{loc}}(I)$. Hence $X(t) = X_0 + \sum_{k\geqslant 0}(X_{k+1} - X_k) \in C(J; \Omega)$ is the solution of (ODE) and is unique by a similar argument as above. □

Proposition 3.1 (Property of the flow). *Assume the hypotheses in Theorem 3.1 with $E = C(I;\mathbb{R}^d)$, $x \in \mathbb{R}^d$ and $v \in L^1_{\mathrm{loc}}(I; \mathrm{Lip}(\mathbb{R}^d;\mathbb{R}^d))$. Then the flow $\psi_t = \psi(t,\cdot) : \mathbb{R}^d \mapsto \mathbb{R}^d$ of the vector field v:*

$$\psi_t(x) = x + \int_0^t v(t', \psi_{t'}(x))\,\mathrm{d}t' \tag{3.16}$$

is a C^1 diffeomorphism on $\mathbb{R}^d$ on the entire time interval I and satisfy

$$\begin{aligned} &\|\nabla\psi_t^{\pm 1}\|_{L^\infty} \leqslant e^{\int_0^t \|\nabla v\|_{L^\infty}}, \\ &\|\nabla\psi_t^{\pm 1} - \mathrm{Id}\,\|_{L^\infty} \leqslant e^{\int_0^t \|\nabla v\|_{L^\infty}} - 1. \end{aligned} \tag{3.17}$$

Proof. (**Exercise**) By Theorem 3.1, for any fixed $x \in \mathbb{R}^d$, there exists a unique solution $X(t;x) \in C(J;\mathbb{R}^d)$ of (ODE) on the time interval $J \subset I$ with $\int_J \|v(t',\cdot)\|_{\mathrm{Lip}}\,\mathrm{d}t' \leqslant \frac{1}{2}$ and a unique continuation argument ensures the unique existence of the solution $X(t;x) \in C(I;\mathbb{R}^d)$. We define $\psi_t(x) = X(t;x)$. Differentiate (3.16) with respect to x yields

$$\partial_{x_j}(\psi_t)^k = \delta_{j,k} + \int_0^t \sum_{l=1}^d \big((\partial_l v^k)(t', \psi_{t'})\big)\big(\partial_{x_j}(\psi_{t'})^l\big)\,\mathrm{d}t'\,.$$

Then the first inequality in (3.17) for ψ_t follows from Gronwall's lemma. Correspondingly we derive the second inequality in (3.17) for ψ_t as follows:

$$\|\nabla\psi_t - \mathrm{Id}\,\|_{L^\infty} \leqslant \int_0^t \|\nabla v\|_{L^\infty} e^{\int_0^{t'} \|\nabla v\|_{L^\infty}}\,\mathrm{d}t' = \int_0^t d e^{\int_0^t \|\nabla v\|_{L^\infty}} = e^{\int_0^t \|\nabla v\|_{L^\infty}} - 1.$$

The inequalities for ψ_t^{-1} follow immediately since

$$\psi_t^{-1}(x) = x + \int_t^0 v(t', \psi_{t'}(\psi_t^{-1}(x)))\,\mathrm{d}t'\,.$$

□

[27.05.2019]

[03.06.2019]

3.1.2 Transport equation

Theorem 3.2. *Let s, p, r satisfy (2.12). Assume the initial data $\rho_0 \in B^s_{p,r}$, the source term $f \in L^1([0,T]; B^s_{p,r})$, and the vector field $v \in L^1([0,T]; \mathrm{Lip}\,) \cap L^q([0,T]; B^M_{\infty,\infty})$ for some $q > 1$ and $M > 1-s$ such that $V(T) < \infty$ with*

$$V(t) = \begin{cases} \int_0^t \|\nabla v\|_{B^{\frac{d}{p}}_{p,\infty} \cap L^\infty}, & \text{if } s < 1 + \frac{d}{p}, \\ \int_0^t \|\nabla v\|_{B^{s-1}_{p,r}}, & \text{if } s > 1 + \frac{d}{p} \text{ or } (s,r) = (1+\frac{d}{p}, 1). \end{cases}$$

Then the equation (T) *has a unique solution* $\rho \in L^\infty([0,T]; B^s_{p,r})$ *such that for a.e.* $t \in [0,T]$,

$$\|\rho(t)\|_{B^s_{p,r}} \leqslant e^{CV(t)}\Big(\|\rho_0\|_{B^s_{p,r}} + \int_0^t e^{-CV(t')}\|f(t')\|_{B^s_{p,r}}\,\mathrm{d}t'\Big). \tag{3.18}$$

Furthermore, if $r < \infty$ *then* $\rho \in C([0,T]; B^s_{p,r})$ *and if* $r = \infty$ *then* $\rho \in C([0,T]; B^{s'}_{p,r})$ *for any* $s' < s$.

Proof. **Step 1 A priori estimate**

Let $\rho \in L^\infty([0,T]; B^s_{p,r})$ be a solution of (T). Let us apply the operator Δ_j to the transport equation (T) to arrive at the equation (2.11). Then by multiplying both sides of the equation by $\mathrm{sgn}(\Delta_j\rho)|\Delta_j\rho|^{p-1}$ and integrating with respect to $x \in \mathbb{R}^d$ yields

$$\begin{aligned}\|\Delta_j\rho(t)\|_{L^p} \leqslant \|\Delta_j\rho_0\|_{L^p} + \int_0^t \big(\|\Delta_j f\|_{L^p} + \|[v\cdot\nabla, \Delta_j]\rho\|_{L^p} \\ + \frac{1}{p}\|\mathrm{div}\, v\|_{L^\infty}\|\Delta_j\rho\|_{L^p}\big)\,\mathrm{d}t'\,.\end{aligned} \tag{3.19}$$

We hence derive from the commutator estimate in Theorem 2.4 that

$$\|\rho(t)\|_{B^s_{p,r}} \leqslant \|\rho_0\|_{B^s_{p,r}} + \int_0^t \big(\|f\|_{B^s_{p,r}} + C(\frac{d}{dt}V)\|\rho\|_{B^s_{p,r}}\big)\,\mathrm{d}t'\,,$$

and the estimate (3.18) follows from the Gronwall's lemma.

Step 2 Approximate solution sequence

Let us regularize the data for the transport equation as follows:

$$\begin{aligned}&\rho_{n,0} = S_n\rho_0 \in B^\infty_{p,r}, \quad f_n = \Phi_n *_t (S_n f) \in C([0,T]; B^\infty_{p,r}),\\ &v_n = \Phi_n *_t (S_n v) \in C_b([0,T]\times\mathbb{R}^d) \text{ such that } \nabla v_n \in C([0,T]; B^\infty_{p,r})\end{aligned}$$

where $\Phi_n = \Phi(t)$ is a mollifier sequence with respect to the time variable. Then the regularized equation

$$\partial_t\rho_n + v_n\cdot\nabla\rho_n = f_n, \quad (\rho_n)|_{t=0} = \rho_{0,n},$$

has a unique solution $\rho_n \in C([0,T]; B^s_{p,r})$:

$$\rho_n(t,x) = \rho_{0,n}(\psi^{-1}_{n,t}(x)) + \int_0^t f_n(t', \psi_{n,t'}(\psi^{-1}_{n,t}(x)))\,\mathrm{d}t'\,,$$

where $\psi_{n,t} : \mathbb{R}^d \mapsto \mathbb{R}^d$ is the flow of the vector field v_n. Then by the estimate (3.18), we have the following estimate for ρ_n:

$$\|\rho_n(t)\|_{B^s_{p,r}} \leqslant e^{CV_n(t)}\Big(\|\rho_{n,0}\|_{B^s_{p,r}} + \int_0^t e^{-CV_n(t')}\|f_n(t')\|_{B^s_{p,r}}\,\mathrm{d}t'\Big),$$

and hence the uniform estimate for $\{\rho_n\}$:

$$\|\rho_n\|_{L^\infty([0,T];B^s_{p,r})} \leqslant e^{CV(T)}\Big(\|\rho_0\|_{B^s_{p,r}} + \int_0^T \|f(t')\|_{B^s_{p,r}}\,\mathrm{d}t'\Big).$$

Step 3: Convergence of the approximate solution sequence

In order to show the convergence of the approximate solution sequence, we prove its compactness with respect to the time variable. To this end, we will show $\partial_t\rho_n$ is uniformly bounded in $L^q([0,T];B^m_{p,\infty})$ for some m small enough, $q>1$ and $T<\infty$. However, as $f\in L^1([0,T];B^s_{p,r})$, we have to first consider $\widetilde{\rho_n} := \rho_n - \int_0^t f_n$ instead.

By the above uniform estimate on $\|\rho_n\|_{L^\infty([0,T];B^s_{p,r})}$ and the assumption $v\in L^q([0,T];B^M_{\infty,\infty})$, $M>1-s$, we derive from the estimates for paraproduct and remainder the uniform bound on $\|v_n\cdot\nabla\rho_n\|_{L^q([0,T];B^m_{p,\infty})}$, $m=\min\{s-1+M, s-1, M\}$:

$$\|v_n\cdot\nabla\rho_n\|_{L^q([0,T];B^m_{p,\infty})} \leqslant C\|v_n\|_{L^q([0,T];B^M_{\infty,\infty})}\|\nabla\rho_n\|_{L^\infty([0,T];B^{s-1}_{p,r})}.$$

Therefore $\widetilde{\rho_n} = \rho_n - \int_0^t f_n = \int_0^t v_n\cdot\nabla\rho_n$ is uniformly bounded in $L^\infty([0,T];B^s_{p,r})$ and $W^{1,q}([0,T];B^m_{p,\infty})$. By the compact embedding $W^{1,q}([0,T])\hookrightarrow\hookrightarrow C([0,T])$ for some $q>1$ and $T<\infty$ and the compactness of the multiplication operator $M_\varphi : B^s_{p,r}\mapsto B^m_{p,\infty}$ via $g\mapsto\varphi g$, $\forall\varphi\in C_0^\infty(\mathbb{R}^d)$, the sequence $\{\varphi\widetilde{\rho_n}\}$ is compact in $C([0,T];B^m_{p,\infty})$ and hence there exists a subsequence (still denoted by (ρ_n, f_n)) such that $\widetilde{\rho_n}\to\widetilde{\rho}$ in $C([0,T];\mathcal{S}')$ and $\varphi\widetilde{\rho_n}\to\varphi\widetilde{\rho}$ in $C([0,T];B^m_{p,\infty})$ and hence in $C([0,T];B^{s'}_{p,r})$, $\forall s'<s$. Thus $\forall s'<s$,

$$\varphi\rho_n = \varphi\widetilde{\rho_n} + \int_0^t \varphi f_n \to \varphi\widetilde{\rho} + \int_0^t \varphi f := \varphi\rho \text{ in } C([0,T];B^{s'}_{p,r}),\quad \forall\varphi\in C_0^\infty(\mathbb{R}^d).$$

Hence the limit ρ satisfies $\partial_t\rho + v\cdot\nabla\rho = f$ at least in the distribution sense. Thus $\rho\in L^\infty([0,T];B^s_{p,r})\cap C([0,T];B^{s'}_{p,r})$ satisfies the estimate (3.18).

In order to show $\rho\in C([0,T];B^s_{p,r})$ if $r<\infty$, we simply use $S_j\rho\in C([0,T];B^\infty_{p,r})$ to approximate ρ in $L^\infty([0,T];B^s_{p,r})$: We estimate the difference $\rho - S_j\rho = \sum_{k\geqslant j}\Delta_k\rho$ by use of the (3.19) as

$$\|\Delta_k\rho\|_{L^p} \leqslant e^{V(t)}\Big(\|\Delta_k\rho_0\|_{L^p} + \int_0^t \|\Delta_k f\|_{L^p} + C\|\rho\|_{L^\infty([0,T];B^s_{p,r})}\int_0^t c_k(t')\,\mathrm{d}t'\Big),$$

for some $c_k(t)\in L^1([0,T];\ell^r)$. Therefore if $r<\infty$ then

$$\Big\|\Big(2^{ks}\big(\|\Delta_k\rho_0\|_{L^p} + \int_0^t \|\Delta_k f\|_{L^p}\big)\Big)_{k\geqslant j}\Big\|_{\ell^r},\quad \Big\|\Big(2^{ks}\int_0^t c_k(t')\Big)_{k\geqslant j}\Big\|_{\ell^r}\to 0,$$

as $j\to\infty$. This implies $\|\rho - S_j\rho\|_{L^\infty([0,T];B^s_{p,r})}\to 0$ as $j\to\infty$ and hence $\rho\in C([0,T];B^s_{p,r})$. □

Remark 3.1. *In particular if $p = r = 2$, then the H^s-regularity of ρ can be transported by the velocity vector field v with $\nabla v \in L^1([0,T]; H^\sigma \cap L^\infty)$, provided with*

$$-\tfrac{d}{2} < s < 1 + \tfrac{d}{2} \text{ and } \sigma = \tfrac{d}{2},$$
$$\text{or } s = \sigma + 1 > 1 + \tfrac{d}{2}.$$

[03.06.2019]

[04.06.2019]

3.2 Navier-Stokes equation

In this subsection we consider the initial value problem for the Navier-Stokes equation

$$\begin{cases} \partial_t u + u \cdot \nabla u - \Delta u + \nabla p = 0, \\ \operatorname{div} u = 0, \\ u|_{t=0} = u_0, \end{cases} \tag{NS}$$

where the time variable $t \geqslant 0$, the space variable $x \in \mathbb{R}^d$, the unknown velocity vector field $u = u(t,x) : [0,\infty) \times \mathbb{R}^d \mapsto \mathbb{R}^d$ and the unknown pressure term $p = p(t,x) : [0,\infty) \times \mathbb{R}^d \mapsto \mathbb{R}$. The Navier-Stokes equation can model the evolution of the incompressible fluid (including liquid and gas), where

- The material derivative $\partial_t + u \cdot \nabla = \partial_t + \sum_{k=1}^d u^k \partial_{x_k}$ corresponds to the transport of the fluid along the velocity vector field u;
- The second order derivative term $-\Delta u = -\sum_{k=1}^d \partial_{x_k x_k} u$ describes the viscosity effect in the fluid;
- The divergence free condition $\operatorname{div} u = \sum_{k=1}^d \partial_{x_k} u^k = 0$ corresponds to the incompressibility of the fluid:

$$\begin{aligned} \frac{d}{dt}(\det(\nabla\varphi)) = tr\Big(adj(\nabla\varphi)\frac{d}{dt}\nabla\varphi\Big) &= tr\Big(adj(\nabla\varphi)\nabla\varphi\nabla u(t,\varphi(t,x))\Big) \\ &= \operatorname{div} u(t,\varphi(t,x)) = 0, \end{aligned}$$

  where $\psi = \psi(t,x) : [0,\infty) \times \mathbb{R}^d \mapsto \mathbb{R}^d$ is the flow of the velocity vector field u: $\frac{d}{dt}\psi(t,x) = u(t,\psi(t,x))$;
- The unknown pressure term $\nabla p = (\partial_{x_j} p)_j$ (not necessarily the physical pressure in the fluid) can be simply viewed as a Lagrangian multiplier associated to the divergence free constraint.

If the solution u is not smooth enough, the term $u \cdot \nabla u$ in (NS) will always be understood as $\mathrm{div}\,(u \otimes u)$ with the matrix $(u \otimes u)_{jk} := (u^j u^k)_{jk}$. Indeed, if u is smooth, then thanks to $\mathrm{div}\, u = 0$,

$$(u \cdot \nabla u) = \sum_{k=1}^{d} u^k (\partial_{x_k} u^j),$$

$$\mathrm{div}\,(u \otimes u) = \sum_{k=1}^{d} \partial_{x_k}(u^j u^k) = \sum_{k=1}^{d} (u^j \partial_k u^k + \partial_k u^j u^k) = u^j \mathrm{div}\, u + u \cdot \nabla u^j = (u \cdot \nabla u).$$

Notice that we need more regularity assumption on u in oder to make sense of the term $u \cdot \nabla u$, than of the term $u \otimes u$.

3.2.1 Weak solutions

It is straightforward to deduce the energy equality for (NS) if the solution is regular enough (say $u \in C^1([0,\infty); (H^\infty(\mathbb{R}^d))^d)$, $\nabla p \in C([0,\infty), (H^\infty(\mathbb{R}^d))^d)$). Let us take $L^2(\mathbb{R}^d)$ inner product between the equation (NS) and u itself, and we calculate the resulting terms one by one:

- $\int_{\mathbb{R}^d} \partial_t u \cdot u = \int_{\mathbb{R}^d} \frac{1}{2}\partial_t(|u|^2) = \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d} |u|^2$,
- $\int_{\mathbb{R}^d} u \cdot \nabla u \cdot u = \int_{\mathbb{R}^d} \frac{1}{2} u \cdot \nabla(|u|^2) = -\int_{\mathbb{R}^d} \frac{1}{2}(\mathrm{div}\, u)|u|^2 = 0$,
- $\int_{\mathbb{R}^d} -\Delta u \cdot u = \int_{\mathbb{R}^d} |\nabla u|^2$;
- $\int_{\mathbb{R}^d} \nabla p \cdot u = -\int_{\mathbb{R}^d} p \cdot \mathrm{div}\, u = 0$.

Thus we arrive at

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d} |u|^2 + \int_{\mathbb{R}^d} |\nabla u|^2 = 0,$$

which implies immediately the energy equality by integration in time:

$$\frac{1}{2}\|u(t)\|^2_{L^2(\mathbb{R}^d)} + \int_0^t \|\nabla u(t')\|^2_{L^2(\mathbb{R}^d)}\,\mathrm{d}t' = \frac{1}{2}\|u_0\|^2_{L^2(\mathbb{R}^d)}. \tag{3.20}$$

Thanks to the above energy estimate, J. Leray proved the global-in-time existence of the weak solution to (NS) in 1934:

Theorem 3.3. *Let u_0 be a divergence-free vector field in $(L^2(\mathbb{R}^d))^d$. Then there exists a weak solution $u \in L^\infty(\mathbb{R}^+; (L^2(\mathbb{R}^d))^d) \cap L^2(\mathbb{R}^+; (\dot{H}^1(\mathbb{R}^d))^d)$ of* (NS) *satisfying the energy inequality:*

$$\frac{1}{2}\|u(t)\|^2_{L^2(\mathbb{R}^d)} + \int_0^t \|\nabla u(t')\|^2_{L^2(\mathbb{R}^d)}\,\mathrm{d}t' \leqslant \frac{1}{2}\|u_0\|^2_{L^2(\mathbb{R}^d)}. \tag{3.21}$$

Here a weak solution $u \in L^2_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}^d)$ *of* (NS) *means that the following equality holds*

$$\begin{aligned}\int_{\mathbb{R}^d} u(t,x)\cdot\varphi(t,x)\,\mathrm{d}x &= \int_0^t\int_{\mathbb{R}^d}(u\cdot\partial_t\varphi + u\otimes u : \nabla\varphi + u\cdot\Delta\varphi)\,\mathrm{d}x\,dt \\ &\quad + \int_{\mathbb{R}^d} u_0(x)\cdot\varphi(0,x)\,\mathrm{d}x\,,\end{aligned}$$

for all $\varphi \in C_0^\infty([0,\infty); (C_0^\infty(\mathbb{R}^d))^d)$ *with* $\operatorname{div}\varphi = 0$.

The proof could be similar as the proof of Theorem 3.2. We construct a sequence of approximated solutions which satisfy the energy estimate (3.21) uniformly. Then the compactness in the time variable as well as a cut-off procedure in the space variable will imply the compactness of the approximated solution sequence in a weaker topology and the limit will satisfy (NS) in the weak sense. However, only the *energy inequality* (3.21) can hold for the weak solution, while the *energy equality* (3.20) holds only under more regularity assumptions in three dimensional case. In particular if $d = 2$, then the above weak solution is unique and satisfies the energy equality (3.20), which was proved by O. Ladyzhenskaya in 1959.

3.2.2 Strong solutions

It is also convenient to rewrite the equation (NS) by eliminating the pressure term ∇p. Indeed, as $\operatorname{div} u = 0$, we introduce the projection operator P:

$$P = (\mathrm{Id} + \nabla(-\Delta)^{-1}\mathrm{div}\,), \text{ i.e. } \widehat{Pv} = \widehat{v^j} - \sum_{k=1}^d \frac{\xi_j\xi_k}{|\xi|^2}\widehat{v^k} = \sum_{k=1}^d(\delta_{j,k} - \frac{\xi_j\xi_k}{|\xi|^2})\widehat{v^k},$$

such that

$$Pu = u, \quad P(\nabla p) = 0.$$

Hence we apply the operator P to the equation (NS) to arrive at

$$\begin{cases}\partial_t u - \Delta u = Q(u,u), \\ u|_{t=0} = u_0,\end{cases} \tag{PNS}$$

where the bilinear operator Q reads as

$$\begin{aligned}Q(v,w) &= -\frac{1}{2}P(\mathrm{div}\,(v\otimes w) + \mathrm{div}\,(w\otimes v)), \\ \text{i.e. } (\widehat{Q(v,w)})^j &= -\frac{1}{2}\sum_{k=1}^d\sum_{l=1}^d(\delta_{j,k} - 1)\frac{i\xi_j\xi_k\xi_l}{|\xi|^2}(\widehat{v^k w^l + v^l w^k}).\end{aligned}$$

We can show the local-in-time well-posedness result of (PNS) in different functional frameworks and here we will follow Kato's L^p approach to show the well-posedness result of (PNS) in $L^3(\mathbb{R}^3)$ in three dimensional case.

Theorem 3.4. *Let $u_0 \in (L^3(\mathbb{R}^3))^3$. Then there exists a positive time T and a unique solution $u \in C([0,T];(L^3(\mathbb{R}^3))^3)$ of the initial value problem* (PNS)*.*

There exists a positive constant c such that if $\|u_0\|_{L^3} \leqslant c$ then T can be chosen as $+\infty$.

Proof. **Step 1 A priori estimate**

Let us take the Fourier transform of the semilinear heat equation (PNS) and then apply Duhamel's formula to arrive at

$$\hat{u}(t,\xi) = e^{-t|\xi|^2}\widehat{u_0}(\xi) - i\sum_{k,l=1}^{d}\int_0^t e^{-(t-t')|\xi|^2}(\delta_{j,k}-1)\frac{\xi_j\xi_k\xi_l}{|\xi|^2}u^k\widehat{(t')u^l}(t')\,\mathrm{d}t'\,.$$

Denote

$$\Gamma^j_{kl}(t,\cdot) = (2\pi)^{-\frac{3}{2}}\mathcal{F}^{-1}\Big(-ie^{-t|\xi|^2}(\delta_{j,k}-1)\frac{\xi_j\xi_k\xi_l}{|\xi|^2}\Big),$$

then the solution u reads as

$$u(t,x) = e^{t\Delta}u_0 + \sum_{k,l=1}^{d}\int_0^t \Gamma_{kl}(t-t',\cdot) * (u^k u^l)(t',\cdot)\,\mathrm{d}t'\,. \tag{3.22}$$

As

$$e^{t\Delta}u_0 = \mathcal{F}^{-1}\big(e^{-t|\xi|^2}\widehat{u_0}(\xi)\big) = (4\pi t)^{-\frac{3}{2}}e^{-\frac{|\cdot|^2}{4t}} * u_0,$$

we apply Young's inequality to derive for $\beta \geqslant 3$

$$\begin{aligned}\|e^{t\Delta}u_0\|_{L^\beta(\mathbb{R}^3)} &\leqslant (4\pi t)^{-\frac{3}{2}}\|e^{-\frac{|x|^2}{4t}}\|_{L^\alpha(\mathbb{R}^3)}\|u_0\|_{L^3(\mathbb{R}^3)}, \text{ with } 1+\frac{1}{\beta} = \frac{1}{\alpha}+\frac{1}{3},\\ &\leqslant Ct^{-\frac{3}{2}+\frac{3}{2\alpha}}\|u_0\|_{L^3(\mathbb{R}^3)} = Ct^{-\frac{1}{2}(1-\frac{3}{\beta})}\|u_0\|_{L^3(\mathbb{R}^3)}.\end{aligned}$$

For any $p \in [1,\infty]$, $T \in (0,\infty)$, we define the norm

$$\|u\|_{K_p(T)} = \sup_{t\in(0,T]} t^{\frac{1}{2}(1-\frac{3}{p})}\|u(t)\|_{L^p(\mathbb{R}^3)}, \tag{3.23}$$

then

$$\|e^{t\Delta}u_0\|_{K_\beta(T)} \leqslant C\|u_0\|_{L^3(\mathbb{R}^3)}, \quad \forall \beta \geqslant 3. \tag{3.24}$$

Now we focus on Γ^j_{kl}. We can rewrite Γ^j_{kl} as

$$\begin{aligned}
\Gamma^j_{kl} &= (2\pi)^{-\frac{3}{2}}(\delta_{j,k}-1)\partial_j\partial_k\partial_l\mathcal{F}^{-1}(e^{-t|\xi|^2}|\xi|^{-2}) \\
&= (2\pi)^{-\frac{3}{2}}(\delta_{j,k}-1)\partial_j\partial_k\partial_l\mathcal{F}^{-1}\Big(\int_t^\infty e^{-t|\xi|^2}\,\mathrm{d}t'\Big) \\
&= (2\pi)^{-\frac{3}{2}}(\delta_{j,k}-1)\partial_j\partial_k\partial_l\int_t^\infty (2t)^{-\frac{3}{2}}e^{-\frac{|x|^2}{4t}}\,\mathrm{d}t' \\
&= (2\pi)^{-\frac{3}{2}}(\delta_{j,k}-1)\int_t^\infty (2t)^{-\frac{3}{2}}(4t)^{-\frac{3}{2}}\big(\partial_j\partial_k\partial_l e^{-|\cdot|^2}\big)(\frac{x}{\sqrt{4t}})\,\mathrm{d}t' \\
&= -\pi^{-\frac{3}{2}}(\delta_{j,k}-1)|x|^{-4}\int_0^{\frac{|x|^2}{4t}} s\big(\partial_j\partial_k\partial_l e^{-|\cdot|^2}\big)(\frac{x}{|x|}\sqrt{s})ds, \quad \text{with } s=\frac{|x|^2}{4t},
\end{aligned}$$

which implies the pointwise bound for Γ:

$$|\Gamma^j_{kl}| \leqslant C\min\{|x|^{-4}, t^{-2}\} \leqslant C(|x|+\sqrt{t})^{-4}.$$

Hence

$$\|\Gamma^j_{kl}(t,\cdot)\|_{L^\alpha(\mathbb{R}^3)} \leqslant C\Big(\int_0^{\sqrt{t}}(t^{-2})^\alpha r^2\,\mathrm{d}r + \int_{\sqrt{t}}^\infty (r^{-4})^\alpha r^2\,\mathrm{d}r\Big)^{\frac{1}{\alpha}} \leqslant Ct^{-2+\frac{3}{2\alpha}}, \quad \forall \alpha\in[1,\infty].$$

Therefore Young's inequality ensures for any $(p,q)\in[1,\infty]^2$ with $\frac{1}{p}+\frac{1}{q}\leqslant 1$,

$$\begin{aligned}
&\Big\|\int_0^t \Gamma_{kl}(t-t',\cdot)*(u^ku^l)(t',\cdot)\,\mathrm{d}t'\Big\|_{L^\beta(\mathbb{R}^3)} \\
&\leqslant C\int_0^t (t-t')^{-2+\frac{3}{2}(1+\frac{1}{\beta}-\frac{1}{p}-\frac{1}{q})}\|u(t')\|_{L^p(\mathbb{R}^3)}\|u(t')\|_{L^q(\mathbb{R}^3)}\,\mathrm{d}t' \\
&\leqslant C\int_0^t (t-t')^{-\frac{1}{2}+\frac{3}{2}(\frac{1}{\beta}-\frac{1}{p}-\frac{1}{q})}(t')^{-1+\frac{3}{2}(\frac{1}{p}+\frac{1}{q})}\,\mathrm{d}t' \\
&\qquad\times\big(\sup_{t'\geqslant 0}(t')^{\frac{1}{2}(1-\frac{3}{p})}\|u(t')\|_{L^p(\mathbb{R}^3)}\big)\big(\sup_{t'\geqslant 0}(t')^{\frac{1}{2}(1-\frac{3}{q})}\|u(t')\|_{L^q(\mathbb{R}^3)}\big).
\end{aligned}$$

If $\frac{1}{3}+\frac{1}{\beta}>\frac{1}{p}+\frac{1}{q}$, we can control the above time integral by

$$\begin{aligned}
&Ct^{-\frac{1}{2}+\frac{3}{2}(\frac{1}{\beta}-\frac{1}{p}-\frac{1}{q})}\int_0^{t/2}(t')^{-1+\frac{3}{2}(\frac{1}{p}+\frac{1}{q})}\,\mathrm{d}t' + Ct^{-1+\frac{3}{2}(\frac{1}{p}+\frac{1}{q})}\int_{t/2}^t (t-t')^{-\frac{1}{2}+\frac{3}{2}(\frac{1}{\beta}-\frac{1}{p}-\frac{1}{q})}\,\mathrm{d}t' \\
&\leqslant Ct^{-\frac{1}{2}+\frac{3}{2\beta}} = Ct^{-\frac{1}{2}(1-\frac{3}{\beta})}.
\end{aligned}$$

Then we have arrived at

$$\begin{aligned}
&\Big\|\textstyle\int_0^t \Gamma_{kl}(t-t',\cdot)*(u^ku^l)(t')\,\mathrm{d}t'\Big\|_{K_\beta(T)} \leqslant C\|u\|_{K_p(T)}\|u\|_{K_q(T)}, \\
&\quad \text{if } \tfrac{1}{\beta}\leqslant\tfrac{1}{p}+\tfrac{1}{q}<\tfrac{1}{3}+\tfrac{1}{\beta},\ \tfrac{1}{p}+\tfrac{1}{q}\leqslant 1.
\end{aligned} \tag{3.25}$$

Step 2 Existence & Uniqueness of the solution in $K_6(T)$

We have established the a priori estimates (3.24)-(3.25) for the solution (3.22) to (PNS) in Step 1. We would like to use the contraction mapping argument to show the existence and uniqueness of the solution in the Banach space

$$K_6(T) = \{u \in C((0,T]; L^6(\mathbb{R}^3)) \mid \|u\|_{K_6(T)} < \infty\}.$$

Indeed, we first rewrite (3.22) into the following form

$$u = a + B(u,u), \quad a := e^{t\Delta}u_0, \quad B(u,u) := \int_0^t \Gamma_{kl}(t-t',\cdot) * (u^k u^l)(t',\cdot)\,\mathrm{d}t'\,.$$

It is easy to see that if $u_0 \in L^3(\mathbb{R}^3)$, then $e^{t\Delta}u_0 \in K_6(T)$ for any $T \in (0,\infty)$. As $\Gamma \in C((0,\infty); L^\alpha(\mathbb{R}^3))$, $\forall \alpha \in [1,\infty)$, the bilinear map

$$B : K_6(T) \times K_6(T) \mapsto K_6(T), \text{ with } \|B(u,v)\|_{K_6(T)} \leqslant C\|u\|_{K_6(T)}\|v\|_{K_6(T)}.$$

For any $u_0 \in L^3(\mathbb{R}^3)$, for any $\varepsilon > 0$, there exists $\varphi \in \mathcal{S}(\mathbb{R}^3)$ such that $\|u_0 - \varphi\|_{L^3(\mathbb{R}^3)} < \varepsilon$. On the other hand, $\|e^{t\Delta}\varphi\|_{L^\infty([0,T];L^6)} \leqslant C\|\varphi\|_{L^6}$. Thus

$$\begin{aligned}\|e^{t\Delta}u_0\|_{K_6(T)} &\leqslant \|e^{t\Delta}(u_0-\varphi)\|_{K_6(T)} + \|e^{t\Delta}\varphi\|_{K_6(T)}\\ &\leqslant C\|u_0-\varphi\|_{L^3} + CT^{\frac{1}{2}(1-\frac{3}{6})}\|\varphi\|_{L^6} \leqslant C\varepsilon + CT^{\frac{1}{4}}\|\varphi\|_{L^6}.\end{aligned}$$

We can choose T sufficiently small (depending on u_0, ε) such that

$$\|e^{t\Delta}u_0\|_{K_6(T)} \leqslant C\varepsilon. \tag{3.26}$$

Therefore for $\varepsilon > 0$, $T > 0$ sufficiently small, we derive from the contraction mapping argument that there exists a unique fixed point u of the map $u \mapsto a + B(u,u)$ in the Banach space $K_6(T)$, with

$$\|u\|_{K_6(T)} \leqslant 2\|e^{t\Delta}u_0\|_{K_6(T)}. \tag{3.27}$$

If $\|u_0\|_{L^3(\mathbb{R}^3)} < c$, then

$$\|e^{t\Delta}u_0\|_{K_6(T)} \leqslant C\|u_0\|_{L^3} \leqslant Cc, \quad \forall T \in (0,\infty).$$

Hence in the small data case that $c > 0$ is sufficiently small, there exists a unique fixed point $u \in K_6(T)$ for any $T \in (0,\infty)$, with $\|u\|_{K_6(T)} \leqslant 2\|e^{t\Delta}u_0\|_{K_6(T)}$.

Step 3 Continuity with values in $L^3(\mathbb{R}^3)$

Although we have showed in Step 2 the existence and the uniqueness of the solution $u \in K_6(T)$, we have to prove further $u \in C([0,T]; L^3)$ and the uniqueness of the solution therein.

Now $u \in K_6(T)$ with $\|u\|_{K_6(T)} \leqslant 2\|e^{t\Delta}u_0\|_{K_6(T)}$ is the known function and we would like to show

$$u = a + B(u,u) \in C([0,T];L^3).$$

Obviously $a = e^{t\Delta}u_0 \in C([0,T];L^3)$. As $u \in K_6(T)$, we infer from the derivation of the estimate (3.25) (with $\beta = 3$) that $B(u,u) \in C((0,T];L^3)$ and for any $t \in (0,T)$,

$$\|B(u,u)\|_{L^\infty([0,t];L^3)} \leqslant C\|u\|_{K_6(t)}^2 \leqslant 4C\|e^{t\Delta}u_0\|_{K_6(t)}^2,$$

where the righthand side tends to zero as $t \to 0^+$. This implies the continuity of $B(u,u)$ at time zero and hence $B(u,u) \in C([0,T];L^3)$.

Step 4 Uniqueness in $C([0,T];L^3(\mathbb{R}^3))$

Let $u, v \in C([0,T];L^3)$ be two solutions to (PNS) and we would like to show $u = v$. To this end, we will use energy estimates in the L^2 functional framework for their difference

$$w = u-v = (e^{t\Delta}u_0+B(u,u))-(e^{t\Delta}u_0+B(v,v)) = B(u,u)-B(v,v) \in C([0,T];L^3).$$

Indeed, by the equation (PNS), w satisfies the equation

$$\begin{cases} \partial_t w - \Delta w = Q(u,u) - Q(v,v) \\ := f = Q(e^{t\Delta}u_0, w) + Q(w, e^{t\Delta}u_0) + Q(B(u,u),w) + Q(w, B(v,v)), \\ w|_{t=0} = 0. \end{cases} \tag{w}$$

By (3.25) with $p = q = 3$, $\beta = 2$ we have

$$\|w\|_{K_2(T)} \leqslant \|B(u,u)\|_{K_2(T)} + \|B(v,v)\|_{K_2(T)} \leqslant C\|u\|_{K_3(T)}^2 + C\|v\|_{K_3(T)}^2,$$

with $\|w\|_{K_2(T)} = \sup_{t\in(0,T]} t^{-\frac14}\|w(t)\|_{L^2}$, and hence $w \in C([0,T];L^2)$.

By Sobolev embedding $\dot H^{\frac12}(\mathbb{R}^3) = \dot B^{\frac12}_{2,2}(\mathbb{R}^3) \hookrightarrow \dot B^0_{3,2}(\mathbb{R}^3) \hookrightarrow L^3(\mathbb{R}^3)$ (by Proposition 2.2 and Proposition 2.3) and $L^{\frac32}(\mathbb{R}^3) \hookrightarrow \dot H^{-\frac12}(\mathbb{R}^3)$ (by duality), we have the following estimate for the bilinear operator $Q(a,b) = -\frac12 P(\mathrm{div}\,(a\otimes b) + \mathrm{div}\,(b\otimes a))$ (noticing the zero-order projection operator $P : \dot H^s \mapsto \dot H^s$):

$$\begin{aligned}\|Q(a,b)\|_{\dot H^{-\frac32}} &\leqslant C\|a\otimes b\|_{\dot H^{-\frac12}} \leqslant C\|a\otimes b\|_{L^{\frac32}} \\ &\leqslant C\min\{\|a\|_{L^3}\|b\|_{L^3}, \|a\|_{L^6}\|b\|_{L^2}, \|a\|_{L^2}\|b\|_{L^6}\}.\end{aligned}$$

Thus $f \in C([0,T];\dot H^{-\frac32})$. We take $\dot H^{-\frac12}$ inner product between the w-equation and w itself to arrive at

$$\frac12\frac{d}{dt}\|w\|_{\dot H^{-\frac12}}^2 + \|\nabla w\|_{\dot H^{-\frac12}}^2 = \langle f, w\rangle_{\dot H^{-\frac32},\dot H^{\frac12}},$$

and hence

$$\frac{1}{2}\|w(t)\|^2_{\dot H^{-\frac12}} + \|\nabla w\|^2_{L^2([0,t];\dot H^{-\frac12})} = \int_0^t \langle f, w\rangle_{\dot H^{-\frac32},\dot H^{\frac12}} \leqslant \frac{1}{2}\|w\|^2_{L^2([0,t];\dot H^{\frac12})} + \frac{1}{2}\|f\|^2_{L^2([0,t];\dot H^{-\frac32})},$$

which implies

$$\|w\|_{L^\infty([0,t];\dot H^{-\frac12})\cap L^2([0,t];\dot H^{\frac12})} \leqslant \|f\|_{L^2([0,t];\dot H^{-\frac32})}, \quad \forall t \in [0,T].$$

Finally we would like to use Gronwall's inequality to deduce $w = 0$ in $L^\infty([0,t];\dot H^{-\frac12})$, at least in small time interval $[0,t]$. To this end, we decompose f into two parts

$$f = f_1 + f_2, \quad f_1 = Q(e^{t\Delta}u_{0,1}, w) + Q(w, e^{t\Delta}u_{0,1}) + Q(B(u,u), w) + Q(w, B(v,v)),$$
$$f_2 = Q(e^{t\Delta}u_{0,2}, w) + Q(w, e^{t\Delta}u_{0,2}), \text{ with } u_0 = u_{0,1} + u_{0,2},$$

and we expect that

$$\|f_1\|_{L^2([0,t];\dot H^{-\frac32})} < \frac{1}{2}\|w\|_{L^2([0,t];\dot H^{\frac12})}, \text{ for small time } t > 0,$$
$$\|f_2\|^2_{L^2([0,t];\dot H^{-\frac32})} \text{ involves low regularity of } w \text{ in the form } C(u_0)\int_0^t \|w\|^2_{\dot H^{-\frac12}}\,\mathrm{d}t'.$$

Indeed, we decompose u_0 into $u_{0,1}$ with small L^3 value: $\|u_{0,1}\|_{L^3} \leqslant c$ and $u_{0,2}$ with regular L^6 value: $u_{0,2} \in L^6$ (e.g. we can simply take $u_{0,2} = S_j u_0$ with sufficiently large j). Then

$$\|f_1\|_{L^2([0,t];\dot H^{-\frac32})} \leqslant C\Big\|\big(\|e^{t\Delta}u_{0,1}\|_{L^3} + \|B(u,u)\|_{K_3(t)} + \|B(v,v)\|_{K_3(t)}\big)\|w\|_{L^3}\Big\|_{L^2([0,t])}$$
$$\leqslant C\big(\|e^{t\Delta}u_{0,1}\|_{L^\infty([0,t];L^3)} + \|B(u,u)\|_{K_3(t)} + \|B(v,v)\|_{K_3(t)}\big)\|w\|_{L^2([0,t];\dot H^{\frac12})} < \frac{1}{2}\|w\|_{L^2([0,t];\dot H^{\frac12})},$$

if c and the time t is chosen small enough and

$$\begin{aligned}\|f_2\|^2_{L^2([0,t];\dot H^{-\frac32})} &\leqslant \int_0^t \|e^{t\Delta}u_{0,2}\|^2_{L^6}\|w\|^2_{L^2} \leqslant C\|u_{0,2}\|^2_{L^6}\int_0^t \|w\|_{\dot H^{-\frac12}}\|w\|_{\dot H^{\frac12}} \\ &\leqslant \frac{1}{8}\|w\|^2_{L^2([0,t];\dot H^{\frac12})} + C\|u_{0,2}\|^4_{L^6}\int_0^t \|w\|^2_{\dot H^{-\frac12}}.\end{aligned}$$

□