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Exercise 1

Let $f \in L^1(\mathbb{R}^d; \mathbb{C})$. Show that its Fourier transform $\hat{f}$ is continuous and satisfies

$$\lim_{|\xi|\to\infty} |\hat{f}(\xi)| = 0.$$

Exercise 2

Let $a \in \mathbb{R}^d$, $f, g \in L^1(\mathbb{R}^d)$ and A be a real invertible $d \times d$ matrix. Show that

$$\mathcal{F}(\tau_a(f)) = e^{ia\cdot\xi}\mathcal{F}(f), \quad \tau_a(f) = f(x+a),$$
$$\mathcal{F}(e^{ia\cdot x} f) = \mathcal{F}(f)(\xi - a),$$
$$\mathcal{F}(f \circ A) = |\det A|^{-1}\mathcal{F}(f \circ A^{-T}),$$
$$\int_{\mathbb{R}^d} f\mathcal{F}(g) = \int_{\mathbb{R}^d} \mathcal{F}(f)g.$$

Exercise 3

Recall that the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ is the set of the smooth functions $f \in C^\infty(\mathbb{R}^d)$ satisfying for any $k \in \mathbb{N}$

$$\|f\|_{k,\mathcal{S}} := \sup_{x\in\mathbb{R}^d,\, |\alpha|\le k} (1+|x|^k)|\partial^\alpha f(x)| < \infty.$$

Show that it is equivalent to say that

$$\mathcal{S}(\mathbb{R}^d) = \{f \in C^\infty(\mathbb{R}^d) \mid \sup_{x\in\mathbb{R}^d} |x^\alpha \partial^\beta f(x)| < \infty, \forall \text{ multiindices } \alpha, \beta\}.$$

Show that

- $\mathcal{S}(\mathbb{R}^d) \subset C^k(\mathbb{R}^d)$ for all $k \in \mathbb{N}$ and $\mathcal{S}(\mathbb{R}^d) \subset L^p(\mathbb{R}^d)$ for all $p \in [1,\infty]$,
- If $f \in \mathcal{S}(\mathbb{R}^d)$, then $x^\alpha f$, $\partial^\alpha f \in \mathcal{S}(\mathbb{R}^d)$ for any multiindex α,
- If $f \in \mathcal{S}(\mathbb{R}^d)$ and $g \in C_b^\infty(\mathbb{R}^d)$, then $fg \in \mathcal{S}(\mathbb{R}^d)$,
- If $f \in \mathcal{S}(\mathbb{R}^d)$ and $g \in \mathcal{D}'(\mathbb{R}^d) = (C_0^\infty(\mathbb{R}^d))'$ with compact support, then $f * g \in \mathcal{S}(\mathbb{R}^d)$,
- If $f, g \in \mathcal{S}(\mathbb{R}^d)$, then $f * g \in \mathcal{S}(\mathbb{R}^d)$,
- The Gaussian function $e^{-\frac{1}{2}|x|^2} \in \mathcal{S}(\mathbb{R}^d)$.

Exercise 4

We define the metric $d(\cdot,\cdot)$ on $\mathcal{S}(\mathbb{R}^d)$:

$$d(f,g) := \sum_{k\in\mathbb{N}} 2^{-k} \frac{\|f-g\|_{k,\mathcal{S}}}{1+\|f-g\|_{k,\mathcal{S}}}.$$

Show that the space $(\mathcal{S}(\mathbb{R}^d), d(\cdot,\cdot))$ is a complete metric space and the space $\mathcal{D}(\mathbb{R}^d) = C_0^\infty(\mathbb{R}^d)$ of smooth compactly supported functions is dense in it. Hence $\mathcal{S}(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$, $\forall p \in [1,\infty)$.

Exercise 5

Let $f \in \mathcal{S}(\mathbb{R}^d)$. Show that the following equalities hold true:

$$\mathcal{F}((\frac{1}{i}\partial_x)^\alpha f) = \xi^\alpha \mathcal{F}(f), \quad \mathcal{F}(x^\alpha f) = (i\partial_\xi)^\alpha \mathcal{F}(f), \quad \forall \text{multiindex } \alpha.$$

Exercise 6

Let $f \in \mathcal{S}(\mathbb{R}^d)$. Show that $e^{-\frac{\varepsilon^2}{2}|x|^2} f \to f$ in $\mathcal{S}(\mathbb{R}^d)$ as $\varepsilon \to 0$, and hence, $(2\pi)^{-\frac{d}{2}}\varepsilon^{-d} e^{-\frac{1}{2\varepsilon^2}|x|^2} * f \to f$ in $\mathcal{S}(\mathbb{R}^d)$ as $\varepsilon \to 0$.

Exercise 7

Let $f, g \in \mathcal{S}(\mathbb{R}^d)$, show that

$$\mathcal{F}(f*g) = (2\pi)^{\frac{d}{2}} \mathcal{F}(f)\mathcal{F}(g), \quad \mathcal{F}(fg) = (2\pi)^{-\frac{d}{2}} \mathcal{F}(f) * \mathcal{F}(g),$$

and

$$\int_{\mathbb{R}^d} f\bar{g} = \int_{\mathbb{R}^d} \hat{f}\bar{\hat{g}}.$$

In particular, $\|f\|_{L^2} = \|\hat{f}\|_{L^2}$.

Exercise 8

Recall that the function $g \in L^1(\mathbb{R}^d)$ defines a tempered distribution $T_g \in \mathcal{S}'(\mathbb{R}^d)$ as

$$T_g(f) = \int_{\mathbb{R}^d} gf, \quad \forall f \in \mathcal{S}(\mathbb{R}^d).$$

Let $\phi \in C^\infty$ with at most polynomial growth, $f \in \mathcal{S}(\mathbb{R}^d)$ and $T \in \mathcal{S}'(\mathbb{R}^d)$. Show that the following equalities hold true:

- The product by $\phi : (\phi T_g)(f) = T_{\phi g}(f)$,
- The derivative $\partial_{x_j} : (\partial_{x_j} T_g)(f) = -T_{\partial_{x_j} g}(f)$,
- The convolution with $f : (T_g * f)(x) = g * f$,
- The Fourier transform $\mathcal{F} : \mathcal{F}(T_g)(f) = T_{\mathcal{F}(g)}(f)$,
- The inverse Fourier transform $\mathcal{F}^{-1}(T_g)(f) = T_{\mathcal{F}^{-1}(g)}(f)$.

Exercise 9

Let $T \in \mathcal{S}'(\mathbb{R}^d)$. Show that, if $\mathcal{F}(T)$ is a polynomial then $\text{Supp}\,(T) = \{0\}$.