Summary term 2019 **Karlsruher Institut für Technologie**
Fourier analysis and its applications to PDEs JProf. Dr. Xian Liao
Exercise sheet 2 M.Sc. Zihui He

Exercise 1

Let $x \in \mathbb{R}$. Show that $\mathcal{F}(e^{-|x|}) = \sqrt{\frac{2}{\pi}} \frac{1}{1+\xi^2}$.

Exercise 2

Let $x \in \mathbb{R}^d$ and $\sigma \in (0, d)$. Show that $\mathcal{F}(|x|^{-\sigma}) = c|\xi|^{\sigma-d}$ for some constant c.

Exercise 3

Let $\chi \in \mathcal{D}(\mathcal{B})$, $\mathcal{B} = \{\xi \in \mathbb{R}^d \,|\, |\xi| \leq \frac{4}{3}\}$ and $\varphi = \chi(\frac{\cdot}{2}) - \chi(\cdot)$. Recall the following operators:

- $\Delta_j u$ with $\widehat{\Delta_j u} = \varphi(2^{-j}\xi)\hat{u}(\xi), \quad j \in \mathbb{N}$,
- $S_j u = \sum_{j' \leq j-1} \Delta_{j'} u$, i.e. $\widehat{S_j u} = \chi(2^{-j}\xi)\hat{u}(\xi), \quad j \in \mathbb{N}$,
- $\dot{\Delta}_j u = \varphi(2^{-j}D)u$ with $\varphi(\widehat{2^{-j}D})u = \varphi(2^{-j}\xi)\hat{u}(\xi), \quad j \in \mathbb{Z}$,
- $\dot{S}_j u = \chi(2^{-j}D)u$ with $\chi(\widehat{2^{-j}D})u = \chi(2^{-j}\xi)\hat{u}(\xi), \quad j \in \mathbb{Z}$.

Recall that the homogeneous Besov norms are defined as

$$\|u\|_{\dot{B}^s_{p,r}} = \left\| \left(2^{js} \|\dot{\Delta}_j u\|_{L^p(\mathbb{R}^d)} \right)_{j\in\mathbb{Z}} \right\|_{\ell^r},$$

where $s \in \mathbb{R}$, $(p, r) \in [1, \infty]^2$.

(1) Show that $|x|^{-\sigma} \in \dot{B}^{d-\sigma}_{1,\infty}(\mathbb{R}^d)$ for $\sigma \in (0, d)$.

(2) Let $u \in \mathcal{S}'(\mathbb{R}^d)$. Show that

$$S_j u \to u \text{ in } \mathcal{S}'(\mathbb{R}^d), \text{ as } j \to \infty.$$

(3) Let $\mathcal{S}_0(\mathbb{R}^d) = \{f \in \mathcal{S}(\mathbb{R}^d) \,|\, \mathrm{Supp}\,(\hat{f}) \cap \{0\} = \emptyset\}$ and $u^R_{M,N} = (\mathrm{Id} - \dot{S}_{-M})\big(\theta(\frac{\cdot}{R})u_N\big), \quad u_N = \sum_{|j|\leq N} \dot{\Delta}_j u$, $R > 0$. Show that $u^R_{M,N} \in \mathcal{S}_0(\mathbb{R}^d)$.

(4) Let $u \in \dot{B}^s_{p,r}(\mathbb{R}^d)$. We take φ and $\tilde{\varphi}$ in the dyadic partition. Show that $\|u\|_{\dot{B}^s_{p,r}} \sim \|u\|_{\widetilde{\dot{B}^s_{p,r}}}$.

Hint: There exists N such that $\mathrm{Supp}\,\varphi(2^{-j}\cdot) \cap \mathrm{Supp}\,\tilde{\varphi}(2^{-j'}\cdot) = \emptyset$ if $|j - j'| \geq N$.

(5) Let $u \in \dot{B}^s_{p,r}(\mathbb{R}^d)$. Show that

$$2^{js}\|\dot{\Delta}_j\big(u(2^N x)\big)\|_{L^p} = 2^{N(s-\frac{d}{p})}\Big(2^{(j-N)s}\|\dot{\Delta}_{j-N}u\|_{L^p}\Big),\ N \geq 0.$$

Exercise 4

Let $\mathcal{B} = \{\xi \in \mathbb{R}^d \,|\, |\xi| \le R\}$ be a ball centered at 0 with radius $R > 0$ and $\mathcal{C} = \{\xi \in \mathbb{R}^d \,|\, 0 < r_1 \le |\xi| \le r_2\}$ be an annulus.

(1) Recall that, there exists a constant C such that the following facts hold for any $t > 0$, $k \in \mathbb{N}$ $p, q \in [1, \infty]$ with $p \le q$ and $u \in L^p(\mathbb{R}^d)$:

$$\begin{aligned}
&\mathrm{Supp}\,(\hat{u}) \subset \mathcal{B} \Rightarrow \sup{}_{|\alpha|=k} \|\partial^\alpha u\|_{L^q} \le C^{k+1} \|u\|_{L^p},\\
&\mathrm{Supp}\,(\hat{u}) \subset \mathcal{C} \Rightarrow C^{-k-1} \|u\|_{L^p} \le \sup{}_{|\alpha|=k} \|\partial^\alpha u\|_{L^p} \le C^{k+1} \|u\|_{L^p},\\
&\mathrm{Supp}\,(\hat{u}) \subset \mathcal{C} \Rightarrow \|e^{t\Delta} u\|_{L^q} \le C e^{-C^{-1}t} \|u\|_{L^p}.
\end{aligned}$$

Show that the following facts hold true for $\lambda > 0$:

$$\begin{aligned}
&\mathrm{Supp}\,(\hat{u}) \subset \lambda\mathcal{B} \Rightarrow \sup{}_{|\alpha|=k} \|\partial^\alpha u\|_{L^q} \le C^{k+1} \lambda^{k+\frac{d}{p}-\frac{d}{q}} \|u\|_{L^p},\\
&\mathrm{Supp}\,(\hat{u}) \subset \lambda\mathcal{C} \Rightarrow C^{-k-1} \lambda^k \|u\|_{L^p} \le \sup{}_{|\alpha|=k} \|\partial^\alpha u\|_{L^p} \le C^{k+1} \lambda^k \|u\|_{L^p},\\
&\mathrm{Supp}\,(\hat{u}) \subset \lambda\mathcal{C} \Rightarrow \|e^{t\Delta} u\|_{L^q} \le C e^{-C^{-1}t\lambda^2} \lambda^{\frac{d}{p}-\frac{d}{q}} \|u\|_{L^p}.
\end{aligned}$$

(2) Show that there exists a constant C such that for any $T > 0$, $\lambda > 0$, $1 \le p \le q \le \infty$, $1 \le b \le a \le \infty$ there holds

$$\mathrm{Supp}\,(\hat{u_0}) \subset \lambda\mathcal{C} \underset{u|_{t=0}=u_0}{\overset{(\partial_t - \mu\Delta)u=0}{\Longrightarrow}} \|u\|_{L^a([0,T];L^q)} \le C(\mu\lambda^2)^{-\frac{1}{a}} \lambda^{\frac{d}{p}-\frac{d}{q}} \|u_0\|_{L^p},$$

and

$$\begin{aligned}
&\mathrm{Supp}\,(\hat{f}(t,\cdot)) \subset \lambda\mathcal{C},\ \forall t \in [0,T]\\
&\underset{u|_{t=0}=0}{\overset{(\partial_t - \mu\Delta)u=f}{\Longrightarrow}} \|u\|_{L^a([0,T];L^q)} \le C(\mu\lambda^2)^{-1+\frac{1}{b}-\frac{1}{a}} \lambda^{\frac{d}{p}-\frac{d}{q}} \|f\|_{L^b([0,T];L^p)}.
\end{aligned}$$

Exercise 5

Let $\varphi \in \mathcal{D}(\mathbb{R}^d \backslash \{0\})$ and $r \in [1, \infty]$. Show that, there exists a constant C such that

$$\|\mathcal{F}^{-1}\big(\varphi e^{-t|\xi|^2}\big)\|_{L^r} \le C \|(1-\Delta)^d (\varphi e^{-t|\xi|^2})\|_{L^1} \le C e^{-C^{-1}t}.$$